Physics 137B

Solutions to Problem Set #3

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Problem 1:

$$H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}kx^2}_{H_0} + \underbrace{\lambda e^{-ax^2}}_{H'}$$

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \quad ; |1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} \, 2\left(\sqrt{\frac{m\omega}{\hbar}}\right) x \, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$E_0^{(1)} = \langle 0|H'|0\rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \lambda e^{-ax^2} e^{-\frac{m\omega}{\hbar}x^2} dx = \lambda \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\left(a+\frac{m\omega}{\hbar}\right)x^2} dx$$

$$= \lambda \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{\pi}{a+\frac{m\omega}{\hbar}}} = \lambda \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \sqrt{\frac{\pi\hbar}{a\hbar+m\omega}} = \lambda \left(\frac{m\omega}{a\hbar+m\omega}\right)^{1/2}$$

$$E_1^{(1)} = \langle 1|H'|1\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} 2\left(\frac{m\omega}{\hbar}\right) \int_{-\infty}^{\infty} \lambda x^2 e^{-\left(a+\frac{m\omega}{\hbar}\right)x^2} dx = \frac{\cancel{2}}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)^{3/2} \frac{\sqrt{\pi}\,\lambda}{\cancel{2}\left(a+\frac{m\omega}{\hbar}\right)^{3/2}}$$

$$= \lambda \left(\frac{m\omega}{\hbar}\right)^{3/2} \left(\frac{\hbar}{a\hbar+m\omega}\right)^{3/2} = \lambda \left(\frac{m\omega}{a\hbar+m\omega}\right)^{3/2}$$

Problem 2:

$$H = \begin{cases} -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{Ze^2}{(4\pi\epsilon_0)2R}\left(\frac{r^2}{R^2} - 3\right) & r \le R \\ -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} & r > R \end{cases}$$

If we take $H_0 = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{(4\pi\epsilon_0)r}$ we see that:

$$H' = \begin{cases} \dfrac{Ze^2}{(4\pi\epsilon_0)2R}\left(\dfrac{r^2}{R^2}-3\right) + \dfrac{Ze^2}{4\pi\epsilon_0}\dfrac{1}{r} & r \leq R \\ 0 & r > R \end{cases}$$

$$\frac{Ze^2}{4\pi\epsilon_0 2R}\left(\frac{r^2}{R^2}-3\right) + \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r} = \frac{Ze^2}{4\pi\epsilon_0}\left[\frac{r^2}{2R^3} - \frac{3R^2}{2R^2} + \frac{1}{r}\right] = \frac{Ze^2}{4\pi\epsilon_0}\left[\frac{r^3 - 3R^2 r + 2R^3}{2R^3 r}\right]$$

$$= \frac{Ze^2}{(4\pi\epsilon_0)2R}\left[\frac{r^2}{R^2} + \frac{2R}{r} - 3\right]$$

(a) We need to compute the first order energy shifts. Notice that hydrogen states are highly degenerate (degeneracy is n^2 for $\psi_{n\ell m}(r,\theta,\phi)$) So we have to diagonalize H' in the degenerate subspaces.

My claim is that H' is already diagonal in the $\psi_{n\ell m}$ basis.

There are two ways to see this:

1) Notice that H' is independent of θ & ϕ, so all off-diagonal matrix elements will include integrals that look like

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) \sin\theta \, d\theta \, d\phi \quad (\ell,m \neq \ell,m' \text{ for off diagonal elements})$$

$= 0$ by orthonormality of Y_ℓ^m

2) L_z commutes with H' since $[L_z, r] = [L_z, f(r)] = 0$ (f any function) & $\psi_{n\ell m}$ are eigenstates of L_z with distinct eigenvalues (in each degenerate subspace). <u>Note</u>: we are ignoring spin here.

$\therefore$ By the theorem proved in discussion section, $\psi_{n\ell m}$ are "good" states (ie. H' is diagonal in this basis)

$$\therefore \Delta E = \langle \psi_{n\ell m}| H' |\psi_{n\ell m}\rangle = \frac{Ze^2}{(4\pi\epsilon_0)(2R)}\int_0^R r^2[R_{n\ell}(r)]^2\left(\frac{r^2}{R^2}+\frac{2R}{r}-3\right)dr \underbrace{\int_0^{2\pi}\int_0^{\pi}|Y_{\ell m}|^2 \sin\theta \, d\theta \, d\phi}_{1}$$

$$= \frac{Ze^2}{(4\pi\epsilon_0)(2R)}\int_0^R r^2(R_{n\ell}(r))^2\left(\frac{r^2}{R^2}+\frac{2R}{r}-3\right)dr$$

(b) $R_{n\ell}(r) \sim R_{n\ell}(0)$

Recall that $R_{n\ell}(r) = \sqrt{\left(\frac{2Z}{na_\mu}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}}\; e^{-rZ/na_\mu}\left(\frac{2rZ}{na_\mu}\right)^\ell L^{2\ell+1}_{n+\ell}\left(\frac{2rZ}{na_\mu}\right)$

Notice that b/c of $\left(\frac{2rZ}{na_\mu}\right)^\ell$ term, $R_{n\ell}(0) = 0$ unless $\ell = 0$

For $\ell = 0$

$$R_{n0}(r) = \sqrt{\left(\frac{2Z}{na_\mu}\right)^3 \frac{(n-1)!}{2n\,(n!)^3}}\; L^1_n(0)$$

Recall that L^1_{n-1} is associated Laguerre polynomial defined by

$$L^{2\ell+1}_{n+\ell}(x) = \sum_{k=0}^{n-\ell-1} (-1)^{k+1} \frac{[(n+\ell)!]^2}{(n-\ell-1-k)!\,(2\ell+1+k)!}\,\frac{x^k}{k!}$$

so $L^1_n(0) = \sum_{k=0}^{n-1} (-1)^{k+1} \frac{[n!]^2}{(n-1-k)!\,(k+1)!}\,\frac{0^k}{k!}$ only the $k=0$ term survives

$$= -\frac{(n!)^2}{(n-1)!}$$

$$|R_{n0}(r)|^2 = +8\left(\frac{Z^3}{n^3a_\mu^3}\right)\frac{(n-1)!}{2n(n!)^3}\left(\frac{(n!)^2}{(n-1)!}\right)^2 = +4\left(\frac{Z^3}{n^3a_\mu^3}\right)\frac{n!}{n\,(n-1)!} = +4\left(\frac{Z^3}{n^3a_\mu^3}\right)\frac{\cancel{n(n-1)!}}{\cancel{n(n-1)!}}$$

$$= +4\left(\frac{Z^3}{n^3a_\mu^3}\right)$$

$$\therefore \Delta E = +2\,\frac{Z^4e^2}{n^3a_\mu^3}\,\frac{1}{4\pi\epsilon_0 R}\left[\int_0^R \frac{r^4}{R^2} + 2Rr - 3r^2\,dr\right]\delta_{\ell 0}$$ ← Recall that if $\ell = 0$, $\Delta E = 0$ since $R_{n\ell}(0) = 0$

$$= \frac{e^2}{4\pi\epsilon_0}\,\frac{Z^4}{n^3a_\mu^3}\left(\frac{2}{R}\right)\Big[\underbrace{\frac{r^5}{5R^2}\Big|_0^R + Rr^2\Big|_0^R - r^3\Big|_0^R}_{\frac{R^3}{5} + R^3 - R^3 = \frac{R^3}{5}}\Big]\delta_{\ell 0}$$

$$= \frac{e^2}{4\pi\epsilon_0}\left(\frac{Z^4}{n^3a_\mu^3}\right)\frac{2}{5}R^2\,\delta_{\ell 0}$$ as desired.

Problem 3

(1) $E_r^{(1)} = \frac{1}{2}(H_{11}' + H_{22}') - (-1)^r \frac{1}{2}\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}$

(2) $\frac{c_{r1}}{c_{r2}} = \frac{-H_{12}'}{H_{11}' - E_r^{(1)}} = -\left(\frac{H_{22}' - E_r^{(1)}}{H_{21}'}\right)$

(3) $|c_{r1}|^2 + |c_{r2}|^2 = 1$, choose phase for c_{r1} s.t. it is real & positive.

Using eqn (2): $c_{r1} = \left(-\frac{H_{12}'}{H_{11}' - E_r^{(1)}}\right) c_{r2}$

$\therefore |c_{r1}|^2 = c_{r1}^* c_{r1} = \left(-\frac{H_{12}^{*\prime}}{(H_{11}' - E_r^{(1)})^*}\right) c_{r2}^* \left(-\frac{H_{12}'}{(H_{11}' - E_r^{(1)})}\right) c_{r2}$

Notice the following: Since H' is a hermitian operator: $H_{11}'^* = H_{11}$ $\quad H_{12}'^* = H_{21}'$

$H_{22}'^* = H_{22}$ $\quad H_{21}'^* = H_{12}'$

$\therefore |c_{r1}|^2 = \frac{H_{21}' H_{12}'}{(H_{11}' - E_r^{(1)})^2} |c_{r2}|^2 \Rightarrow |c_{r2}|^2 = \frac{(H_{11}' - E_r^{(1)})^2}{H_{21}' H_{12}'} |c_{r1}|^2$

Plug into (3):

$|c_{r1}|^2 + |c_{r2}|^2 = \left[1 + \frac{(H_{11}' - E_r^{(1)})^2}{H_{21}' H_{12}'}\right] |c_{r1}|^2 = 1 \Rightarrow |c_{r1}|^2 = \frac{H_{21}' H_{12}'}{H_{21}' H_{12}' + (H_{11}' - E_r^{(1)})^2}$

What is $H_{21}' H_{12}'$?

Recall where all these equations came from. $\det \begin{pmatrix} H_{11}' - E_r^{(1)} & H_{12}' \\ H_{21}' & H_{22}' - E_r^{(1)} \end{pmatrix} = 0$

$\Rightarrow (H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)}) - H_{12}' H_{21}' = 0$

$\therefore H_{12}' H_{21}' = (H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)})$

$\therefore |c_{r1}|^2 = \frac{(H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)})}{(H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)}) + (H_{11}' - E_r^{(1)})^2} = \frac{H_{22}' - E_r^{(1)}}{H_{22}' - E_r^{(1)} + H_{11}' - E_r^{(1)}} =$

Compute numerator & denominator:

$$\therefore H_{22}' - E_r^{(1)} = \frac{1}{2}\left(H_{22}' - H_{11}'\right) + (-1)^r \frac{1}{2}\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}$$

$$H_{22}' - E_r^{(1)} + H_{11}' - E_r^{(1)} = H_{22}' + H_{11}' - 2E_r^{(1)} = H_{22}' + H_{11}' - \left[(H_{11}' + H_{22}') - (-1)^r\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}\right]$$

$$= (-1)^r\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}$$

$$\therefore |c_{1r}|^2 = \frac{\frac{1}{2}\left(H_{22}' - H_{11}'\right) + (-1)^r \frac{1}{2}\left[(H_{11}' - H_{22}')^2 - 4|H_{12}'|^2\right]^{1/2}}{(-1)^r\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}}$$

$$= \frac{1}{2} + \frac{1}{2}(-1)^r \frac{(H_{22}' - H_{11}')}{\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}}$$

so if we choose the phase s.t. c_{r1} is real & positive.

$$c_{r1} = \frac{1}{\sqrt{2}}\left[1 + (-1)^r \frac{H_{22}' - H_{11}'}{\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}}\right]^{1/2} = \frac{1}{\sqrt{2}}\left[1 - (-1)^r \frac{H_{11}' - H_{22}'}{\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}}\right]^{1/2}$$

Now, let's work on c_{r2}:

$c_{r2} = -\left(\frac{H_{11}' - E_r^{(1)}}{H_{12}}\right) c_{r1}$; H_{11}', $E_r^{(1)}$ & c_{r1} are all real numbers

HOWEVER H_{12}' is complex.

ie $H_{12}' = re^{i\theta}$ where r is a real number.

So let's write c_{r2} in the same form as H_{12}', ie. $\boxed{c_{r2} = r'e^{i\theta'}}$

$$c_{r2} = -\underbrace{\frac{|H_{12}'|}{H_{12}'}}_{\text{just a phase}}\underbrace{\left[\frac{H_{11}' - E_r^{(1)}}{|H_{12}'|}\right]c_{r1}}_{\text{real}} \qquad \text{where } |H_{12}'| = |r|$$

since $\frac{|H_{12}'|}{H_{12}} = \frac{|r|}{re^{i\theta}} = \pm e^{-i\theta}$

$= r'e^{i\theta'}$; so $r' = \frac{H_{11}' - E_r'}{|H_{12}'|} c_{r1}$ and $e^{i\theta'} = \pm e^{i\theta}$

let us compute r' by computing $|c_{r2}|^2 = \left(\frac{(H_{11}' - E_r^{(1)})^2}{|H_{12}'|^2}\right)|c_{r1}|^2$

$$|c_{r2}|^2 = \frac{(H_{11}' - E_r^{(1)})^2}{(H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)})} |c_{r1}|^2 = \frac{H_{11}' - E_r^{(1)}}{H_{22}' - E_r^{(1)}} |c_{r1}|^2$$

$$= \frac{H_{11}' - E_r^{(1)}}{\cancel{H_{22}' - E_r^{(1)}}} \frac{\cancel{H_{22}' - E_r^{(1)}}}{H_{22}' - E_r^{(1)} + H_{11}' - E_r^{(1)}} = \frac{H_{11}' - E_r^{(1)}}{H_{22}' + H_{11}' - 2E_r^{(1)}}$$

$$H_{11}' - E_r^{(1)} = \frac{1}{2}(H_{11}' - H_{22}') + (-1)^r \frac{1}{2} \left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2 \right]^{1/2}$$

$$\therefore |c_{r2}|^2 = \frac{1}{2} + \frac{1}{2}(-1)^r \frac{H_{11}' - H_{22}'}{\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}}$$

$$\therefore c_{r2} = -\frac{|H_{12}'|}{H_{12}'} (\pm) \frac{1}{\sqrt{2}} \left[1 + (-1)^r \frac{H_{11}' - H_{22}'}{\left[(H_{11}' - H_{22}')^2 + 4|H_{12}'|^2\right]^{1/2}} \right]^{1/2}$$

The final step is to figure out which r is the positive root & which r is the negative root.

Recall that I wrote $c_{r2} = -\frac{|H_{12}'|}{H_{12}'} \left[\frac{H_{11}' - E_r^{(1)}}{|H_{12}'|} \right] c_{r1}$

but using eqn (2) I could also write it as $c_{r2} = -\frac{H_{21}'}{|H_{21}|} \left[\frac{|H_{21}|}{H_{22}' - E_r^{(1)}} \right] c_{r1}$

Notice that if $H_{12}' = re^{i\theta}$ then $H_{21}' = re^{-i\theta}$ $\therefore$ $\frac{H_{21}'}{|H_{21}'|} = \frac{re^{i\theta}}{|r|} = \pm e^{-i\theta} = \frac{|H_{12}'|}{|H_{12}|}$

And since $|H_{12}'|$ & $|H_{21}'|$ are positive, the only way these equations can equal each other is that if $H_{11}' - E_r^{(1)}$ & $H_{22}' - E_r^{(1)}$ have the same sign.

$$\frac{H_{11}' - E_r^{(1)}}{|H_{12}'|} = \frac{|H_{21}'|}{H_{22}' - E_r^{(1)}} \Rightarrow (H_{11}' - E_r^{(1)})(H_{22}' - E_r^{(1)}) = |H_{12}'||H_{21}'| > 0$$

Here the argument gets a difficult to follow but bare with me for a second.

$$H_{11}' - E_r^{(1)} = \overbrace{\frac{1}{2}(H_{11}' - H_{22}')}^{B} + \overbrace{(-1)^r \frac{1}{2}[\cdots]^{1/2}}^{A} = A + B$$

$$H_{22}' - E_r^{(1)} = \frac{1}{2}(H_{22}' - H_{11}') + (-1)^r \frac{1}{2}[\cdots]^{1/2} = A - B$$

both must have the same sign. Given this you can show (try the 4 cases where A & B are positive or negative) that $|A| > |B|$

$\therefore$ If $B>0$ $\begin{cases} H_{11}'-E_r^{(1)} < 0 & \text{for } r=1 \\ H_{11}'-E_r^{(1)} > 0 & \text{for } r=2 \end{cases}$

If $B<0$ $\begin{cases} H_{11}'-E_r^{(1)} < 0 & \text{for } r=1 \\ H_{11}'-E_r^{(1)} > 0 & \text{for } r=2 \end{cases}$

$$\therefore \quad H_{11}'-E_r^{(1)} = (-1)^r \left| H_{11}-E_r^{(1)} \right|$$

$$\therefore \quad c_{r_2} = -\frac{|H_{12}'|}{H_{12}} (-1)^r \frac{1}{\sqrt{2}} \left[1+(-1)^r \frac{H_{11}'-H_{22}'}{\left[(H_{11}'-H_{22}')^2+4|H_{12}'|^2 \right]^{1/2}} \right]^{1/2}$$

$$= (-1)^{r+1} \frac{|H_{12}'|}{H_{12}} \frac{1}{\sqrt{2}} \left[1+(-1)^r \frac{H_{11}'-H_{22}'}{\left[(H_{11}'-H_{22}')^2+4|H_{12}'|^2 \right]^{1/2}} \right]^{1/2}$$

Problem 4

$$\psi_{np}(x,y) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{p\pi y}{L}\right) \quad , \quad E_{np} = E_1 (n^2+p^2)$$

- The ground state is $n=1$, $p=1$ & it is non-degenerate

$$E_{11}^{(1)} = \langle 1,1 | H' | 1,1 \rangle = 10^{-3} E_1 \frac{4}{L^2} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx \underbrace{\int_0^L \sin^2\left(\frac{\pi y}{L}\right) dy}_{\frac{L}{2} \text{ by normalization}}$$

$$= 10^{-3} E_1 \frac{2}{L} \int_0^L \underbrace{\sin^2\left(\frac{\pi x}{L}\right)}_{\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= 10^{-3} E_1 \frac{2}{L} \left[\frac{1}{2} \underbrace{\int_0^L \sin\left(\frac{\pi x}{L}\right) dx}_{\frac{2L}{\pi}} - \frac{1}{2} \underbrace{\int_0^L \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx}_{\text{Use } \int_0^L \sin\left(\frac{a\pi x}{L}\right)\cos\left(\frac{b\pi x}{L}\right) dx = \frac{L}{\pi} \frac{a}{a^2-b^2}\left(1-(-1)^{a+b}\right)} \right]$$

$$= 10^{-3} E_1 \frac{2}{L} \left[\frac{L}{\pi} - \frac{1}{2} \frac{L}{\pi} \frac{1}{1-4} \underbrace{\left(1-(-1)^5\right)}_{-2} \right]$$

$$= 10^{-3} E_1 \frac{2}{L} \left[\frac{L}{\pi} - \frac{L}{3\pi} \right] = 10^{-3} E_1 \frac{2}{L} \left[\frac{2L}{3\pi} \right] = \boxed{10^{-3} \frac{4}{3}\pi E_1}$$

• Now let's look at the doubly degenerate eigenstates. We need ~~to~~ to diagonalize H'.

$$H' = \begin{pmatrix} \langle n,p|H'|n,p\rangle & \langle n,p|H'|p,n\rangle \\ \langle p,n|H'|n,p\rangle & \langle p,n|H'|p,n\rangle \end{pmatrix}$$

My claim is that H' is already diagonal in this basis. You can see this two ways.

- <u>Clever way</u>: Notice that these states are momentum eigenstates. In particular they are eigenstates of P_y with non degenerate eigenvalues.

  $P_y|n,p\rangle = -\frac{p^2}{L^2}|n,p\rangle$ & $P_y|p,n\rangle = -\frac{n^2}{L^2}|n,p\rangle$

  Also $[P_y, H'] = 0$ since $[P_y, x] = 0$.

  ∴ By the "theorem" we talked about in discussion, this basis is a "good" basis.

- <u>Direct computation</u>: Look at the off diagonal elements:

$$\langle n,p|H'|p,n\rangle = \frac{4}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{p\pi x}{L}\right)\left(10^{-3}E_1\sin\left(\frac{\pi x}{L}\right)\right)\underbrace{\int_0^L \sin\left(\frac{p\pi y}{L}\right)\sin\left(\frac{n\pi y}{L}\right)dy}_{\substack{=0 \text{ by orthogonality} \\ \text{of } \sin\left(\frac{n\pi y}{L}\right) \text{ \& } \\ \sin\left(\frac{p\pi y}{L}\right).}}$$

The other off diagonal element is very similar.

$$\therefore H' = \begin{pmatrix} \langle n,p|H'|n,p\rangle & 0 \\ 0 & \langle p,n|H'|p,n\rangle \end{pmatrix} \text{ already diagonal!}$$

So, let's compute these corrections:

$$\langle n,p|H'|n,p\rangle = \frac{4}{L^2}10^{-3}E_1\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx\underbrace{\int_0^L \sin^2\left(\frac{p\pi y}{L}\right)dy}_{\frac{L}{2}}$$

$$= \frac{2}{L}10^{-3}E_1\int_0^L\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right]\sin\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2}{L} 10^{-3} E_1 \left[\underbrace{\frac{1}{2} \int_0^L \sin\left(\frac{\pi x}{L}\right) dx}_{\frac{2L}{\pi}} - \frac{1}{2} \int_0^L \cos\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx \right]$$

$$= \frac{2}{L} 10^{-3} E_1 \left[\frac{L}{\pi} - \frac{L}{2\pi} \frac{1}{1-4n^2} \left(1-(-1)^{1+2n}\right) \right] = 10^{-3} E_1 \left[\frac{2}{\pi} + \frac{1}{\pi} \frac{1}{4n^2-1} \left(1-(-1)^{2n+1}\right) \right]$$

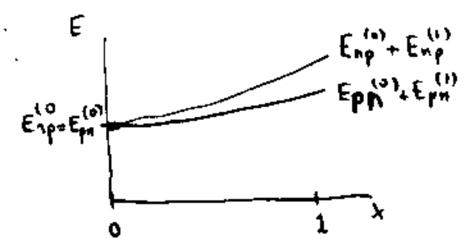
notice that $2n+1$ is odd for any integer n.

$$\therefore \; E_{np}^{(1)} = 10^{-3} E_1 \left[\frac{2}{\pi} + \frac{2}{\pi} \frac{1}{4n^2-1} \right]$$

$$E_{pn}^{(1)} = \frac{4}{L^2} 10^{-3} E_1 \int_0^L \sin^2\left(\frac{p\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx \int_0^L \sin^2\left(\frac{n\pi y}{L}\right) dy$$

The math is the same except $n \to p$.

$$\therefore \; E_{pn}^{(1)} = 10^{-3} E_1 \left[\frac{2}{\pi} + \frac{2}{\pi} \frac{1}{4p^2-1} \right]$$



Problem 5

In this problem we have two masses separated by a distance a. Apart from the net motion of the dumbell as a whole (i.e the center of mass motion), the only thing these two particles are allowed to do is to rotate. As a result, the energy is given by rotational kinetic energy, which classically is just $\frac{L^2}{2I}$ (a rotational analog of $\frac{p^2}{2m}$ for translational kinetic energy)

$I = \frac{1}{2} m a^2$

As a result, $\hat{H} = \dfrac{\hat{L}^2}{2I} \rightarrow E_\ell = \dfrac{\hbar^2 \ell(\ell+1)}{2I}$ with eigenstates $|\ell, m_\ell\rangle$

so there is $2\ell+1$ degeneracy for any given ℓ.

The given perturbation is $H' = -\vec{d}\cdot\vec{E}$ where $\vec{d}$ is the dipole moment ($\vec{d} = \sum_i q_i \vec{r}_i$ in EM). Choose coordinates so that $\vec{E}$ (electric field) is in the $\hat{z}$ direction. $\therefore H' = -dE\cos\theta$

Now, we have to diagonalize each deg. subspace and show that there is no energy shift.

So we have to evaluate matrix elements that look like:

$$\langle \ell, m | H' | \ell, m' \rangle = -\int_0^{2\pi}\int_0^{\pi} Y_\ell^{*m} Y_\ell^{m} \, dE\cos\theta \, \sin\theta \, d\theta \, d\phi$$

(same ℓ, different m_ℓ).

Now, we have to use a parity argument.

My claim is that H' is an odd parity operator. This is easiest to see in cartesian coordinates $H' = -\vec{d}\cdot\vec{E} = -d_z E = -eEz$ (look at the picture).

$\vec{d} = \sum e\vec{r}_i = e\vec{r}_1 - e\vec{r}_2$

$= e(\vec{r}_1 - \vec{r}_2)$

$\therefore \vec{d}\cdot\vec{E} = eE(\vec{r}_1 - \vec{r}_2)\cdot\hat{z}$

Recall that z is an odd parity operator, and that the parity of Y_ℓ^m is $(-1)^\ell$. Since in each degenerate subspace ℓ is fixed, the parity of $|Y_\ell^m|^2$ is always even. Integral of an odd parity function (odd times even is odd) over all space is 0.

(You could also think about the parity of H' in spherical coordinates. Recall that under parity $r \rightarrow r$, $\theta \rightarrow \pi - \theta$, $\phi \rightarrow \phi + \pi$

$\therefore PH' = -dE\cos(\pi-\theta) = dE\cos\theta = -H'$ ← odd parity)