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Solutions to PS #4

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## Problem 1

(a) $H' = -\vec{\mu}\cdot\vec{B} = -\dfrac{e}{2m_e c} B L_z$

$H_0 = \dfrac{L^2}{2I}$ $\quad |\psi_\ell^{(0)}\rangle = |\ell, m\rangle \leftarrow 2\ell+1$ degenerate; $E_\ell = \dfrac{\hbar^2 \ell(\ell+1)}{2I}$

Notice that $|\ell, m\rangle$ basis is a good basis since $L_z$ commutes with $H'$ $([L_z, L_z]=0)$ & $|\ell, m\rangle$ are non degenerate eigenvectors of it.

$\therefore$ $E_m^{(1)} = \langle \ell, m | H' | \ell, m\rangle = \langle \ell, m| -\dfrac{e}{2m_e c} B L_z |\ell, m\rangle = -\dfrac{eB}{2m_e c}\hbar m$

$E = E_\ell^{(0)} + E_m^{(1)} = \dfrac{\hbar^2 \ell(\ell+1)}{2I} - \dfrac{eB}{2m_e c}\hbar m$

(b) P states are states with $\ell = 1$. Notice that by the same argument as part a $|n, \ell, m\rangle$ basis is a good basis. (I have ignored spin here. If you want to include it recall that $\mu = \dfrac{e}{m_e c} S$ for spin. $\therefore$ You must add a term to $H' = -\dfrac{e}{m_e c} S_z$. Notice that with both of these terms $|n \ell m_\ell\rangle \times |s, m_s\rangle$ is a good basis).

For $\ell = 1$ we have $|n,1,1\rangle$, $|n,1,0\rangle$, $|n,1,-1\rangle$

$E_1^{(1)} = \langle n,1,1| -\dfrac{e}{2m_e c} B L_z |n,1,1\rangle = -\dfrac{eB\hbar}{2m_e c}$

$E_0^{(1)} = \langle n,1,0| \dfrac{eB}{2m_e c} L_z |n,1,0\rangle = 0$

$E_{-1}^{(1)} = \langle n,1,-1| -\dfrac{eB}{2m_e c} L_z |n,1,-1\rangle = +\dfrac{eB\hbar}{2m_e c}$

$\therefore$ $E_n \rightarrow \begin{cases} E_n + eB\hbar/2m_e c \\ E_n \\ E_n - eB\hbar/2m_e c \end{cases}$ for $\ell = 1$.

## Problem 2

$H'$ for spin orbit coupling is proportional to $\vec{L}\cdot\vec{S}$

$$H' \sim \vec{L}\cdot\vec{S} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$$

$$[H', \vec{J}] = \frac{1}{2}\begin{pmatrix} [J^2-L^2-S^2, J_x] \\ [J^2-L^2-S^2, J_y] \\ [J^2-L^2-S^2, J_z] \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} [L^2+S^2, J_x] \\ [L^2+S^2, J_y] \\ [L^2+S^2, J_z] \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} [L^2+S^2, L_x+S_x] \\ [L^2+S^2, L_y+S_y] \\ [L^2+S^2, L_z+S_z] \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} [L^2, L_x] + [L^2, S_x] + [S^2, L_x] + [S^2, S_x] \\ [L^2, L_y] + [L^2, S_y] + [S^2, L_y] + [S^2, S_y] \\ [L^2, L_z] + [L^2, S_z] + [S^2, L_z] + [S^2, S_z] \end{pmatrix} = 0$$

(Any component of $S$ commutes w/ $L$ since they act on different spaces)

We also need to show that $\vec{J}$ commutes with the rest of the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{e}{4\pi\epsilon_0 r}$$

, Notice that $[H_0, \vec{S}] = 0$ since $\vec{S}$ has no spatial dependence.

$$[H_0, \vec{L}] \sim [p^2, \vec{L}] + \left[\frac{1}{r}, \vec{L}\right]$$

- $$[p^2, \vec{L}] = \begin{pmatrix} [p_x^2+p_y^2+p_z^2, L_x] \\ [p_x^2+p_y^2+p_z^2, L_y] \\ [p_x^2+p_y^2+p_z^2, L_z] \end{pmatrix} = 0.$$

You can check the following commutators:

$[L_z, p_x] = i\hbar p_y$ , $[L_z, p_y] = -i\hbar p_x$ , $[L_z, p_z] = 0$

$[L_y, p_z] = i\hbar p_x$ , $[L_y, p_x] = -i\hbar p_z$ , $[L_y, p_y] = 0$

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- Instead of looking at $\frac{1}{r}$ you can look at $r^2$:

$$[\vec{L}, r^2] = \begin{pmatrix} L_x, x^2+y^2+z^2 \\ L_y, x^2+y^2+z^2 \\ L_z, x^2+y^2+z^2 \end{pmatrix} = 0$$

You can check the following commutators:

$[L_z, x] = i\hbar y$ , $[L_z, y] = -i\hbar x$ , $[L_z, z] = 0$

$[L_y, z] = i\hbar x$ , $[L_y, x] = -i\hbar z$ , $[L_y, y] = 0$

$[L_x, y] = i\hbar z$ , $[L_x, z] = -i\hbar y$ , $[L_x, x] = 0$

$\therefore$ Any function of $r^2$ (for example $r = \sqrt{r^2}$) commutes w/ $\vec{L}$

$\therefore$ $[H_0, \vec{L}] = 0$ $\quad\therefore$ $[H_0, \vec{J}] = 0$

$\therefore$ $\vec{J}$ is conserved.

For $\vec{L}$ & $\vec{S}$, we already saw that they commute with $H_0$.

Let's check $H'$:

$$[H',\vec{L}] = \begin{pmatrix} [L_xS_x + L_yS_y + L_zS_z , L_x] \\ [L_xS_x + L_yS_y + L_zS_z , L_y] \\ [L_xS_x + L_yS_y + L_zS_z , L_z] \end{pmatrix} = \begin{pmatrix} [L_yS_y, L_x] + [L_zS_z, L_x] \\ [L_xS_x, L_y] + [L_zS_z, L_y] \\ [L_xS_x, L_z] + [L_yS_y, L_z] \end{pmatrix}$$

- $[L_yS_y, L_x] = L_y \underbrace{[S_y, \cancel{L_x}]}_{0} + [L_y, L_x]S_y = -i\hbar L_zS_y$
- $[L_zS_z, L_x] = L_z \underbrace{[S_z, \cancel{L_x}]}_{0} + [L_z, L_x]S_z = +i\hbar L_yS_z$
- $[L_xS_x, L_y] = i\hbar L_zS_x$ ; $[L_zS_z, L_y] = -i\hbar L_xS_z$ ; $[L_xS_x, L_z] = -i\hbar L_yS_x$
- $[L_yS_y, L_z] = i\hbar L_xS_y$

$$\therefore (H',\vec{L}) = \begin{pmatrix} i\hbar\,(L_yS_z - L_zS_y) \\ i\hbar\,(L_zS_x - L_xS_z) \\ i\hbar\,(L_xS_y - L_yS_x) \end{pmatrix} = i\hbar\,(\vec{L}\times\vec{S}) \neq 0$$

$\therefore$ $\vec{L}$ is not conserved.

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<u>Elegant way</u>: $[H',\vec{L}] = [\vec{L}\cdot\vec{S}, \vec{L}]$ ; $[\vec{L}\cdot\vec{S},\vec{L}]_i = [L_jS_j, L_i] = L_j \underbrace{[S_j, \cancel{L_i}]}_{0} + [L_j L_i]S_j$

$$= -i\hbar\varepsilon_{ijk}L_kS_j = i\hbar\varepsilon_{ikj}L_kS_j = i\hbar\,(L\times S)_i$$

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Similarly $[H',\vec{S}] = [\vec{L}\cdot\vec{S}, \vec{S}]$; $[\vec{L}\cdot\vec{S},\vec{S}]_i = [L_jS_j, S_i] = L_j[S_j, S_i] + \underbrace{[L_j, \cancel{S_i}]}_{0}S_j$

$$= L_j(-i\hbar\varepsilon_{ijk}S_k) = -i\hbar\,\varepsilon_{ijk}L_jS_k$$

$$= -i\hbar\,(L\times S)_i \neq 0$$

$\therefore$ $\vec{S}$ is not conserved.

# Problem 3

$$E^{(1)}_{hyperfine} = \frac{\mu_0 g e^2}{3\pi m_p m_e a^3}\langle \vec{S}_p \cdot \vec{S}_e \rangle \sim \frac{g}{m_p m_e a^3}$$

(Note: Use this equation, not 6.92 since there has been an implicit bohr radius substitution & one of the $m_e$ is not $m_e$ but the reduced mass)

$\Delta E_{HYDROGEN} = 5.88\times10^{-6}$ eV ; bohr radius → use reduced mass
the $m_e$ & $m_p$ in the denominator come from magnetic moments & should stay the masses of the objects.

(a) Muonic Hydrogen:

$M_\mu = 207\, m_e$

Recall that $a_\mu \sim \frac{1}{\mu}$ ← reduced mass ; $\mu = \frac{m_\mu m_p}{m_\mu + m_p} = \frac{m_\mu}{1 + \frac{m_\mu}{m_p}} = \frac{207\, m_e}{1 + 207\frac{m_e}{m_p}} = 186\, m_e$

$a \sim \frac{1}{m_e}$ $\quad a_\mu \sim \frac{1}{\mu} \Rightarrow a_\mu = \frac{1}{186} a$

$$\Delta E = \frac{\mu_0 g e^2}{3\pi m_p m_\mu a_\mu^3}(\text{Triplet} - \text{Singlet}) = \Delta E_{HYDROGEN}\underbrace{\left(\frac{1}{207}\right)}_{\frac{m_e}{m_\mu}}\underbrace{(186)^3}_{\left(\frac{a}{a_\mu}\right)^3} = 5.88\times10^{-6}\left(\frac{1}{207}\right)(186)^3$$

$$= \boxed{0.183\text{ eV}}$$

(b) Positronium:

g of proton = 5.59
g of positron = 2
$m_p \to m_e$

$\mu = \frac{m_e m_{positron}}{m_e + m_{positron}} = \frac{m_e^2}{2m_e} = \frac{1}{2} m_e \Rightarrow a_\mu = 2a$

$$\therefore \Delta E = \Delta E_{HYDROGEN}\underbrace{\left(\frac{m_p}{m_e}\right)}_{\frac{m_p}{m_{positron}}}\underbrace{\left(\frac{2}{5.59}\right)}_{\text{change in } g}\underbrace{\left(\frac{1}{2}\right)^3}_{\text{change in } a_\mu} = 5.88\times10^{-6}\left(\frac{1.67\times10^{-27}}{9.11\times10^{-31}}\right)\left(\frac{2}{5.59}\right)\left(\frac{1}{8}\right)$$

$$= \boxed{4.82\times10^{-4}\text{ eV}}$$

(c) Muonium:

g of proton = 5.59
g of muon = 2
$m_p \to m_\mu$

$\mu = \frac{m_e m_\mu}{m_e + m_\mu} = \frac{207 m_e^2}{208 m_e} = \frac{207}{208} m_e \Rightarrow a_\mu = \frac{208}{207} a$

$$\Delta E = \Delta E_{HYDROGEN}\left(\frac{m_p}{m_\mu}\right)\left(\frac{2}{5.59}\right)\left(\frac{207}{208}\right)^3 = 5.88\times10^{-6}\left(\frac{2}{5.59}\right)\left(\frac{1.67\times10^{-27}}{207\times9.11\times10^{-31}}\right)\left(\frac{207}{208}\right)^3 = \boxed{1.84\times10^{-5}\text{ eV}}$$

# Problem 4

Deuteron has spin 1
Electron has spin $\frac{1}{2}$ $\Big\}$ $S_{TOT} = \frac{3}{2}$ or $\frac{1}{2}$

$$\mu_d = \frac{g_d e}{2m_d}\vec{S}_d \qquad m_d = m_p + m_n \qquad g_d = 1.71$$

$$\therefore E_{hf}^{(1)} = \frac{\mu_0 g_d e^2}{3 m_d m_e a_\mu^3}\langle \vec{S}_d \cdot \vec{S}_e \rangle$$

$$\vec{S}_d \cdot \vec{S}_e = \frac{1}{2}\left(S_{TOT}^2 - S_d^2 - S_e^2\right)$$

$$\langle S_e^2 \rangle = \frac{3}{4}\hbar^2$$

$$\langle S_d^2 \rangle = \hbar^2 1(1+1) = 2\hbar^2$$

$$\langle S_{TOT}^2 \rangle = \begin{cases} \hbar^2 \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}\hbar^2 & \text{for } S_{TOT} = \frac{1}{2} \\ \hbar^2 \frac{3}{2}\left(1+\frac{3}{2}\right) = \frac{15}{4}\hbar^2 & \text{for } S_{TOT} = \frac{3}{2} \end{cases}$$

$$\therefore \frac{1}{2}\langle S_{TOT}^2 - S_d^2 - S_e^2 \rangle = \begin{cases} -\hbar^2 & \text{for } S_{TOT} = 1/2 \\ \frac{1}{2}\hbar^2 & \text{for } S_{TOT} = 3/2 \end{cases}$$

$$\therefore E_{hf}^{(1)} = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\begin{cases} -\hbar^2 \\ \frac{1}{2}\hbar^2 \end{cases}$$

$$\Delta E^{(1)} = \left(\frac{3}{2}\right)\frac{\mu_0 g_d e^2}{3\pi m_d m_e a_\mu^3} = \left(\frac{3}{2}\right)\Delta E_{HYDROGEN}\left(\frac{m_p}{m_d}\right)\left(\frac{g_d}{g_p}\right)\left(\frac{a}{a_\mu}\right)^3$$

$m_d \simeq m_p + m_n \simeq 2m_p$

$$\mu = \frac{m_e m_d}{m_d + m_e} = \frac{2 m_e m_p}{2m_p + m_e} \approx \frac{2 m_e m_p}{2m_p} = m_e \quad \therefore a_\mu \approx a_0.$$

$$\therefore \Delta E^{(1)} = \left(\frac{3}{2}\right)\Delta E^{(1)}_{HYDROGEN}\underbrace{\left(\frac{1.71}{5.6}\right)}_{\frac{g_d}{g_p}}\underbrace{\frac{1}{2}}_{\approx \frac{m_p}{m_d}} \quad ; \quad \lambda_d = \frac{c h}{\Delta E^{(1)}} = \left(\frac{5.6}{1.71}\right)(2)\left(\frac{2}{3}\right)\lambda_H$$

$$= \frac{5.6}{1.71}\,\frac{4}{3}\underbrace{\lambda_H}_{21\text{cm}} = 92\text{ cm}$$