Physics 137 B, Spring 2004

Solutions to PS #10

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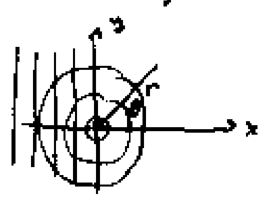
Problem 1: Griffiths 11.2

1-D: In one dimension, there is no such thing as a spherical wave (or you can think of it as the spherical wave being the same thing as a plane wave but going in the opposite direction).

You have already seen scattering wavefunctions in 1D last semester

$$\psi(x) = A\{e^{ikx} + f\, e^{-ikx}\}$$

2-D: The picture is



Incoming wave is still a plane wave e^{ikx}

The scattering wave must be of the form $\frac{e^{ikr}}{\sqrt{r}}$ ← polar coordinates r.

Why? We need it to be a spherical wave so it must $\sim \frac{e^{ikr}}{g(r)}$

We also need to conserve probability: Prob. that the incident particle traveling at speed v, passes through infinitesimal length $d\sigma$ (in 3D $d\sigma$ would be an area) in time dt is

$$P = |\psi_{incident}|^2 (v\,dt)\, d\sigma \quad \left(\text{Notice that } P \text{ is unitless since } |\psi|^2 \sim \frac{1}{\text{Area}},\ (v\,dt)\,d\sigma = \text{length} \times \text{length} = \text{Area}\right)$$

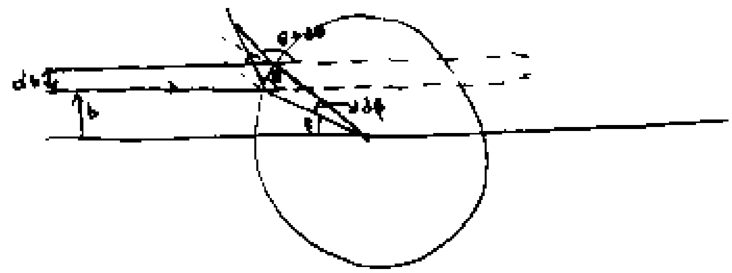
This must be equal to the probability that the particle later emerges into the corresponding angle $r\,d\theta$.

$$P = |\psi_{scattered}|^2 (v\,dt)\, r\, d\theta$$

$$\psi_{incident} = A e^{ikx} \Rightarrow |\psi_{incident}|^2 = |A|^2,\ |\psi_{scattered}|^2 = |A|^2 |f(\theta)|^2 \frac{1}{|g(r)|^2}$$

$$\Rightarrow \frac{d\sigma}{d\theta} = \frac{|f(\theta)|^2 r}{|g(r)|^2} \quad \therefore \text{for } \frac{d\sigma}{d\theta} = |f(\theta)|^2 \Rightarrow g(r) = \sqrt{r} \Rightarrow \psi(r,\theta) = A\left\{e^{ikx} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right\}$$

Problem 2



From the picture we see that # of particles scattered/time w/ scattering angle b/w θ & $\theta + d\theta$ = # of incident particles/time w/ impact parameter b & $b+db$

$$= F\, 2\pi b\, db$$

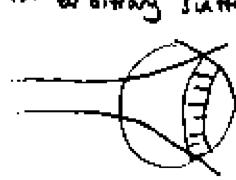
↑ Flux

( Area $= db\,(2\pi b)$)

$$= N\, d\Omega = N \sin\theta\, d\theta\, d\phi$$

$$= N\, 2\pi \sin\theta\, d\theta$$

(azimuthal symmetry)

(For arbitrary scattering  see Griffiths pg 354)

Recall that $\dfrac{d\sigma}{d\Omega} = \dfrac{N}{F}$

$$N\, d\Omega = N\, 2\pi \sin\theta\, d\theta = F\, 2\pi b\, db$$

$$\Rightarrow N = \frac{Fb}{\sin\theta}\left|\frac{db}{d\theta}\right|$$ ← to make sure it is a positive #

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\cancel{F} b}{\sin\theta}\left|\frac{db}{d\theta}\right|\frac{1}{\cancel{F}} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Looking at Figure 13.6 we see that $\sin\beta = \dfrac{b}{a} \Rightarrow b = a\sin\beta$

We also see that $2\beta + \theta = \pi \Rightarrow \beta = \dfrac{\pi - \theta}{2}$

$$\Rightarrow b(\theta) = a \sin\left(\frac{\pi-\theta}{2}\right) = a\cos\left(\frac{\theta}{2}\right)$$

$$\left|\frac{db}{d\theta}\right| = a\left|-\frac{1}{2}\sin\left(\frac{\theta}{2}\right)\right| = \frac{a}{2}\sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\,\frac{a}{2}\sin\left(\frac{\theta}{2}\right) = \frac{a^2\cos(\theta/2)\sin(\theta/2)}{2\sin\theta} = \frac{a^2}{4} \qquad \left(2\cos\theta/2\,\sin\theta/2 = \sin\theta\right)$$

Problem 3

(a) $R = r_0 A^{1/3}$

For $\ell > kR = \frac{pR}{\hbar} = p\frac{r_0 A^{1/3}}{\hbar}$ scattering is negligible.

$\ell = 0$ scattering (s-wave) will dominate when $kR < 1$

$\Rightarrow p\frac{r_0 A^{1/3}}{\hbar} < 1 \quad \Rightarrow p < \frac{\hbar}{r_0 A^{1/3}}$ $\qquad$ $A = 197$ for Gold

$\therefore p < \frac{1.05\times10^{-34}}{1.1\times10^{-15}(197)^{1/3}} = 1.6\times10^{-20}\ \mathrm{kg}\frac{m}{s}$

(b) $p = 1\frac{MeV}{c} = 1.6\times10^{-13}\frac{J}{MeV}\,\frac{1\,MeV}{3\times10^{8}\,m/s} = 5.3\times10^{-22}\ \mathrm{kg}\frac{m}{s}$

So considering only s-wave scattering is justified.

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left|\frac{j_\ell(kR)}{h_\ell^{(1)}(kR)}\right|^2 \approx \frac{4\pi}{k^2}\left|\frac{j_0(kR)}{h_0^{(1)}(kR)}\right|^2$$

$ka \ll 1$ $\qquad j_0(kR) \approx 1$

$$h_0^{(1)}(kR) = j_0(kR) + i n_0(kR) \approx 1 + i\frac{1}{kR} \approx i\frac{1}{kR}$$

$$\frac{j_0(kR)}{h_0^{(1)}(kR)} = \frac{1}{i\frac{1}{kR}} = -i\,kR \quad\Rightarrow\quad \left|\frac{j_0(kR)}{h_0^{(1)}(kR)}\right|^2 = (kR)^2$$

$$\therefore \sigma = \frac{4\pi}{k^2}(kR)^2 = 4\pi R^2 = 4\pi\left(1.1\times10^{-15}(197)^{1/3}\right)^2 = 5.2\times10^{-28}\,m^2$$

(c) p-wave $\quad \sigma \approx \frac{4\pi}{k^2}(3)\left|\frac{j_1(kR)}{h_1^{(1)}(kR)}\right|^2$

$$\frac{j_1(kR)}{h_1^{(1)}(kR)} \approx -i\frac{j_1(kR)}{n_1(kR)} = -i\frac{kR}{(2\ell+1)!!}\left[\frac{(kR)^2}{(2\ell-1)!!}\right] = \frac{-i\,(kR)^3}{1\cdot3\cdot1} = -i\frac{(kR)^3}{3}$$

$$\therefore \sigma_p = \frac{4\pi}{k^2}(3)\frac{(kR)^6}{9} = 12\pi k^4 R^6$$

$$\frac{\sigma_p}{\sigma_s} = \frac{12\pi k^4 R^6}{4\pi R^2} = 3(kR)^4 \sim O^4(kR) \quad (\text{remember that } ka \ll 1)$$

$$kR = \frac{p}{\hbar}R = 3\times10^{-2} \quad \sim (kR)^4 \approx 1\times10^{-6}$$

$\therefore \sigma_p$ is 6 orders of magnitude smaller than σ_s.

Problem 4

$$f(\theta) = \frac{1}{k}\left(e^{ika}\sin ka + 3i\,e^{2ika}\cos\theta\right)$$

Recall that $f(\theta) = \sum_{\ell=0}^{\infty} f_\ell(k)\,P_\ell(\cos\theta)$ Eqn 13.75

(a) $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

for s-wave scattering $f(\theta) = f_0(k)\,1$

Comparing this to the expression given we see that $f_0(k) = \frac{1}{k}e^{ika}\sin ka$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{\sin^2 ka}{k^2}$$

(b) $\frac{d\sigma}{d\Omega} = \frac{N}{F} \Rightarrow N = F\frac{d\sigma(\theta)}{d\Omega} \equiv$ # of particles scattered into direction θ/time/solid angle

We want # of particles / second $= N\,d\Omega$

$$\therefore N\,d\Omega = F\frac{d\sigma(\theta)}{d\Omega}\,d\Omega$$

$F = 10^{14}\,cm^{-2}/s$. We are told that $d\Omega = 4\pi\times10^{-3}$ about the forward direction

$\Rightarrow \theta = 0.$

$\therefore$ We want $F\frac{d\sigma(0)}{d\Omega}\,d\Omega$

$\frac{d\sigma(\theta)}{d\Omega} = |f(\theta)|^2$ here we don't assume s-wave scattering.

$$|f(\theta)|^2 = \frac{1}{k^2}\left[e^{-ika}\sin(ka) + 3i\,e^{-2ika}\cos\theta\right]\left[e^{ika}\sin(ka) + 3i\,e^{2ika}\cos\theta\right]$$

$$= \frac{1}{k^2}\left[\sin^2(ka) - 3i\,e^{i(ka-2ka)}\sin(ka)\cos\theta + 3i\,e^{i(2ka-ka)}\sin(ka)\cos\theta + 9\cos^2\theta\right]$$

$$= \frac{1}{k^2}\left[\sin^2(ka) + \left[-3i\left(\underbrace{\cos(-ka)}_{\cos ka} + i\underbrace{\sin(-ka)}_{-\sin(ka)}\right) + 3i\left(\cos(ka) + i\sin(ka)\right)\right]\sin ka\cos\theta + 9\cos^2\theta\right]$$

$$= \frac{1}{k^2}\left[\sin^2(ka) - 6\sin^2(ka)\cos\theta + 9\cos^2\theta\right]$$

$$\therefore N\,d\Omega = F\frac{d\sigma(\theta)}{d\Omega}\,d\Omega = \frac{F}{k^2}\left[\sin^2(ka) - 6\sin^2(ka) + 9\right]d\Omega$$

$$= \frac{F}{k^2}\left[9 - 5\sin^2(ka)\right]d\Omega$$

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2\times1.6\times10^{-27}\times1.3\times1.6\times10^{-19}}}{1.05\times10^{-34}} = 2.51\times10^{11}\,\frac{1}{m}$$

$$\therefore F = 10^{14}\,cm^{-2}/s = 10^{18}\,m^{-2}/s$$

$$N\,d\Omega = \frac{10^{18}\,m^{-2}/s}{(2.51\times10^{11})^2\,m^{-2}}\left[9 - 5\sin^2(2.51\times10^{11}\cdot2\times10^{-15})\right]4\pi\times10^{-3}$$

$$= 2\times10^{-7}\left[9 - 1.26\times10^{-6}\right] = 1.8\times10^{-6}\ sec^{-1}$$