Physics 137 B Spring 2004

Solutions to PS #5

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Problem 1

(a) $\phi_0(x) = (c^2 - x^2)^2 \qquad |x| < c$

$\qquad\quad = 0 \qquad\qquad\quad |x| \geq c$

We will assume c is real for this problem

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

$$\langle H \rangle = \frac{\langle \phi_0 | H | \phi_0 \rangle}{\langle \phi_0 | \phi_0 \rangle}$$

$$\langle \phi_0 | \phi_0 \rangle = \int_{-\infty}^{\infty} (c^2 - x^2)^4 \, dx = \int_{-c}^{c} (c^2 - x^2)^4 \, dx = \int_{-c}^{c} (c^4 - 2c^2x^2 + x^4)(c^4 - 2c^2x^2 + x^4) \, dx$$

$$= \int_{-c}^{c} c^8 - 4c^6x^2 + c^4x^4 + 4c^4x^4 - 2c^2x^6 + x^4c^4 - 2c^2x^6 + x^8 \, dx$$

$$= \int_{-c}^{c} c^8 - 4c^6x^2 + 6c^4x^4 - 4c^2x^6 + x^8 \, dx$$

$$= c^8 x - \frac{4}{3}c^6x^3 + \frac{1}{5}(6c^4)x^5 - \frac{4}{7}c^2x^7 + \frac{x^9}{9}\Big|_{-c}^{c}$$

$$= c^8(2c) - \frac{4}{3}c^6(2c^3) + \frac{6}{5}c^4(2c^5) - \frac{4}{7}c^2(2c^7) + \frac{(2c^9)}{9}$$

$$= 2c^9 - \frac{8}{3}c^9 + \frac{12}{5}c^9 - \frac{8}{7}c^9 + 2\frac{c^9}{9}$$

$$= \frac{256}{315}c^9$$

$$\langle \phi_0 | H | \phi_0 \rangle = -\frac{\hbar^2}{2m}\int_{-c}^{c} (c^2 - x^2)^2 \frac{d^2}{dx^2}(c^2 - x^2)^2 \, dx + \frac{1}{2}m\omega^2 \int_{-c}^{c} x^2 (c^2 - x^2)^4 \, dx$$

$$= -\frac{\hbar^2}{2m}\int_{-c}^{c} (c^2 - x^2)^2 (12x^2 - 4c^2) \, dx + \frac{1}{2}m\omega^2 \int_{-c}^{c} x^2 (c^8 - 4c^6x^2 + 6c^4x^4 - 4c^2x^6 + x^8) \, dx$$

$$= -\frac{\hbar^2}{2m}\int_{-c}^{c} 12x^6 - 28c^2x^4 + 20x^2c^4 - 4c^6 \, dx + \frac{1}{2}m\omega^2 \left[\frac{c^8}{3}x^3 - \frac{4c^6}{5}x^5 + \frac{6c^4}{7}x^7 - 4c^2\frac{x^9}{9} + \frac{x^{11}}{11} \right]_{-c}^{c}$$

$$= -\frac{\hbar^2}{2m}\left[\frac{12}{7}x^7 - \frac{28c^2x^5}{5} + \frac{20x^3c^4}{3} - 4c^6x\right]\Bigg|_{-c}^{c} + \frac{1}{2}m\omega^2\left[\frac{c^8}{3}(2c^3) - \frac{4c^6}{5}(2c^5) + \frac{6c^4}{7}(2c^7) - \frac{4c^2}{9}(2c^9) + \frac{(2c^{11})}{11}\right]$$

$$= -\frac{\hbar^2}{2m}\left[\frac{12}{7}(2c^7) - \frac{28}{5}c^2(2c^5) + \frac{20}{3}c^4(2c^3) - 4c^6(2c)\right] + \frac{1}{2}m\omega^2\left(\frac{2}{3}c^{11} - \frac{8}{5}c^{11} + \frac{12}{7}c^{11} - \frac{8}{9}c^{11} + \frac{2}{11}c^{11}\right)$$

$$= -\frac{\hbar^2}{2m}\left[\left(\frac{24}{7} - \frac{56}{5} + \frac{40}{3} - 8\right)c^7\right] + \frac{1}{2}m\omega^2\left(\frac{256}{3465}c^{11}\right)$$

$$= -\frac{\hbar^2}{2m}\left(-\frac{256}{105}c^7\right) + \frac{1}{2}m\omega^2\left(\frac{256}{3465}c^{11}\right)$$

$$\therefore \langle H\rangle = \frac{+\frac{\hbar^2}{2m}\left(\frac{256}{105}\right)c^7 + \frac{1}{2}m\omega^2\left(\frac{256}{3465}\right)c^{11}}{\frac{256}{315}c^9} = \frac{3\hbar^2}{2m}c^{-2} + \frac{1}{22}m\omega^2c^2$$

$$\frac{\partial\langle H\rangle}{\partial c} = -\frac{3\hbar^2}{m}c^{-3} + \frac{1}{11}m\omega^2 c = 0 \quad \Rightarrow \frac{1}{11}m\omega^2 c = \frac{3\hbar^2}{mc^3}$$

$$\Rightarrow c^4 = \frac{33\hbar^2}{m^2\omega^2} \Rightarrow c = \pm\left(\frac{33\hbar^2}{m^2\omega^2}\right)^{1/4}$$

(The other two roots are imaginary)

$$\langle H\rangle(c) = \frac{3\hbar^2}{2m}\left(\frac{m^2\omega^2}{33\hbar^2}\right)^{1/2} + \frac{1}{22}m\omega^2\left(\frac{33\hbar^2}{m^2\omega^2}\right)^{1/2}$$

$$= \frac{3\hbar^2}{2m}\frac{m\omega}{\hbar}\left(\frac{1}{33}\right)^{1/2} + \frac{1}{22}\frac{m\omega^2\hbar}{m\omega}(33)^{1/2} = \frac{3}{2}\hbar\omega\left(\frac{1}{33}\right)^{1/2} + \frac{(33)^{1/2}}{22}\hbar\omega$$

$$= \sqrt{\frac{3}{11}}\,\hbar\omega = 0.522233\,\hbar\omega \quad \left(\text{very close to the real answer } \frac{\hbar\omega}{2}\right)$$

(b) $\phi_1(x) = x\,\phi(x)$ is an odd function. Since the harmonic oscillator potential is even we know that the ground state is an even function

$\therefore \langle \phi_1 | \psi_0 \rangle = 0.$

$\uparrow$ real ground state of SHO

$\therefore \phi_1$ will give us an upper bound for the 1st excited state.

$$\langle H\rangle = \frac{\langle \phi_1|H|\phi_1\rangle}{\langle \phi_1|\phi_1\rangle}$$

$$\langle \phi_1|\phi_1\rangle = \int_{-c}^{c} x^2\left(c^8 - 4c^6x^2 + 6c^4x^4 - 4c^2x^6 + x^8\right)dx$$ ← we did this integral in part (a)

$$= \frac{256}{3465}c^{11}$$

$$\langle \phi_1|H|\phi_1\rangle = -\frac{\hbar^2}{2m}\int_{-c}^{c} x\left(c^2-x^2\right)^2\frac{d^2}{dx^2}\left(x\left(c^2-x^2\right)^2\right)dx + \frac{1}{2}m\omega^2\int_{-c}^{c} x^4\left(c^6 - 4c^6x^2 + 6c^4x^4 - 4c^2x^6 + x^8\right)dx$$

$$= \frac{-\hbar^2}{2m}\int_{c}^{c} x\left(c^2-x^2\right)^2\left(20x^3 - 12c^2x\right)dx + \frac{1}{2}m\omega^2\left[\frac{c^6}{5}(2c^5) - \frac{4c^6}{7}(2c^7) + \frac{6c^4}{9}(2c^9) - \frac{4c^2}{11}(2c^{11} + (2c^{13})\right]$$

$$= -\frac{\hbar^2}{2m}\int_{c}^{c} 20x^8 - 52c^2x^6 + 44c^4x^4 - 12c^6x^2\,dx + \frac{1}{2}m\omega^2\left(\frac{256}{15015}c^{13}\right)$$

$$= -\frac{\hbar^2}{2m}\left(-\frac{256}{315}c^9\right) + \frac{1}{2}m\omega^2\left(\frac{256}{15015}c^{13}\right)$$

$$\therefore \langle H\rangle = \frac{\frac{\hbar^2}{2m}\left(\frac{256}{315}\right)c^9 + \frac{1}{2}m\omega^2\left(\frac{256}{15015}\right)c^{13}}{\frac{256}{3465}c^{11}} = \frac{11\hbar^2}{2m}c^{-2} + \frac{3}{26}m\omega^2c^2$$

$$\frac{\partial\langle H\rangle}{\partial c} = -\frac{11\hbar^2}{m}c^{-3} + \frac{3}{13}m\omega^2c = 0 \Rightarrow \frac{3}{13}m\omega^2c = \frac{11}{m}\hbar^2c^{-3}$$

$$\Rightarrow c^4 = \left(\frac{143}{3}\frac{\hbar^2}{m^2\omega^2}\right) \Rightarrow c = \pm\left(\frac{143}{3}\frac{\hbar^2}{m^2\omega^2}\right)^{1/4}$$

$$\langle H\rangle(c) = \frac{11}{2}\frac{\hbar^2}{m}\left(\frac{3}{143}\frac{m^2\omega^2}{\hbar^2}\right)^{1/2} + \frac{3}{26}m\omega^2\left(\frac{143}{3}\frac{\hbar^2}{m^2\omega^2}\right)^{1/2} = \frac{11}{2}\hbar\omega\left(\frac{3}{143}\right)^{1/2} + \frac{3}{26}\hbar\omega\left(\frac{143}{3}\right)^{1/2}$$

$$= \sqrt{\frac{33}{13}}\,\hbar\omega = 1.59326\,\hbar\omega$$ (the correct answer is $\frac{3}{2}\hbar\omega = 1.5\hbar\omega$).

Problem 2

(a) We have 2 neutrons & 1 proton in an infinite square well.

Neutrons & protons are distinguishable since they have different masses.
But the two neutrons are not distinguishable so we must antisymmetrize their wavefunctions.

- What is the ground state energy? The lowest energy we can get is if all 3 particles are in the ground state. We can accomplish this by using a symmetric spatial and an antisymmetric spin wave function for the neutrons.

$$\therefore\ E_{ground} = \frac{\pi^2\hbar^2}{2m_n L^2} + \frac{\pi^2\hbar^2}{2m_n L^2} + \frac{\pi^2\hbar^2}{2m_p L^2} = \frac{\pi^2\hbar^2}{L^2}\left(\frac{1}{m_n} + \frac{1}{2m_p}\right) = \left(\frac{\pi^2\hbar^2}{2L^2 m_p}(3) \text{ if you set } m_n = m_p\right)$$

What is the wave function:

For the neutrons: $\psi(x_1,x_2)\chi(s_1,s_2) = \left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right) + \frac{2}{L}\sin\left(\frac{\pi x_2}{L}\right)\sin\frac{\pi x_1}{L}\right]\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ ← singlet antisymmetric

$$= \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

For the proton: $\psi(x_3)\chi(s_3) = \begin{cases}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x_3}{L}\right)|\uparrow\rangle \\ \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x_3}{L}\right)|\downarrow\rangle\end{cases}$ ← both of these are allowed.

So the total wave function is:

$$\psi(x_1,x_2,x_3)\chi(s_1,s_2,s_3) = \begin{cases}\left(\frac{2}{L}\right)^{3/2}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{\pi x_3}{L}\right)\right]\frac{1}{\sqrt{2}}\left[|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right]|\uparrow_3\rangle \\ \left(\frac{2}{L}\right)^{3/2}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{\pi x_3}{L}\right)\right]\frac{1}{\sqrt{2}}\left[|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right]|\downarrow_3\rangle\end{cases}$$

$\therefore$ 2 fold degenerate

- <u>1st excited state</u>: First we have to decide whether a proton or a neutron should go in the 1st excited state. Notice that $m_n = 939.5\text{ MeV} > m_p = 938.3\text{M}$

The expressions for the energy are

$$E = \frac{\pi^2\hbar^2}{2m_n L^2} + \frac{\pi^2\hbar^2}{2m_n L^2} + \frac{4\pi^2\hbar^2}{2m_p L^2} = \frac{\pi^2\hbar^2}{2L^2}\left[\frac{2}{m_n} + \frac{4}{m_p}\right]$$ ← proton in the 1st excited state

or

$$E = \frac{\pi^2\hbar^2}{2m_n L^2} + \frac{4\pi^2\hbar^2}{2m_n L^2} + \frac{\pi^2\hbar^2}{2m_p L^2} = \frac{\pi^2\hbar^2}{2L^2}\left[\frac{1}{m_n} + \frac{4}{m_n} + \frac{1}{m_p}\right] = \frac{\pi^2\hbar^2}{2L^2}\left[\frac{5}{m_n} + \frac{1}{m_p}\right]$$

$$\frac{2}{m_n} + \frac{4}{m_p} = 0.00639\ (\text{MeV})^{-1}$$

$$\frac{5}{m_n} + \frac{1}{m_p} = 0.005323\ (\text{MeV})^{-1}$$ ← smaller in 1st excited state

$\therefore E_{1^{st} \text{ excited state}} = \frac{\pi^2\hbar^2}{2L^2}\left(\frac{5}{m_n} + \frac{1}{m_p}\right)$

We need to put one of the neutrons in the 1^{st} excited state. Allowed states are:
(Note to save some writing $\psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$, $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$, etc):

$$\psi(x_1,x_2)\chi(s_1,s_2) = \begin{cases} \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\right)\frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right) \\ \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)\right)\left\{\begin{matrix} |\uparrow_1\uparrow_2\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle\right) \\ |\downarrow_1\downarrow_2\rangle \end{matrix}\right\} \text{3 symmetric spin states.} \end{cases}$$

$$\therefore \psi(\underset{n_1}{x_1}, \underset{n_2}{x_2}, \underset{p}{x_3})\chi(s_1,s_2,s_3) = \begin{cases} \psi(x_1,x_2)\chi(s_1,s_2)\,\psi_1(x_3)\,|\uparrow_3\rangle \\ \psi(x_1,x_2)\chi(s_1,s_2)\,\psi_1(x_3)\,|\downarrow_3\rangle \end{cases}$$

Let's count the degeneracy. There are 4 possibilities for $\psi(x_1,x_2)\chi(s_1,s_2)$ & 2 possible ways of combining $\psi(x_1,x_2)\chi(s_1,s_2)$ with the proton wave function.
$\therefore$ The degeneracy is $4 \times 2 = 8$.

<u>NOTE</u>: If you set $m_p = m_n$, then the answer would change in the following manner: $E_{1^{st} \text{ excited state}} = \frac{\pi^2\hbar^2}{2m_pL^2}(6) = \frac{\pi^2\hbar^2}{2m_nL^2}(6)$

There would be two additional states allowed where the neutrons are in the ground state:

$$\psi(x_1,x_2)\chi(s_1,s_2) = \psi_1(x_1)\psi_1(x_2)\frac{1}{\sqrt{2}}\left(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\right)$$

$$\therefore \psi(x_1,x_2,x_3)\chi(s_1,s_2) = \begin{cases} \psi(x_1,x_2)\chi(s_1,s_2)\,\psi_2(x_3)\,|\uparrow_3\rangle \\ \psi(x_1,x_2)\chi(s_1,s_2)\,\psi_2(x_3)\,|\downarrow_3\rangle \end{cases}$$

The degeneracy would be $8 + 2 = 10$.

(b) Now we have $2\pi^0$ & one π^+ (all of them have spin 0).
$\therefore$ We need to symmetrize the π^0 wavefunctions. First notice that since they have spin 0, the spin stays 0 ($s_{TOT} = s_1 + s_2, \ldots, |s_1 - s_2| = 0$). So we don't have to worry about spin wave functions.

• Ground state: All 3 particles in the ground state. ∴ $E_g = \frac{\pi^2\hbar^2}{L^2}\left(\frac{1}{m_{\pi^0}} + \frac{1}{2m_{\pi^+}}\right)$

$\left(\text{if you set } m_{\pi^0} = m_{\pi^+} \Rightarrow E_g = \frac{3\pi^2\hbar^2}{2m_\pi L^2}\right)$

$\Psi(x_1,x_2,x_3) = \psi_1(x_1)\psi_1(x_2)\psi_1(x_3)$ ← degeneracy = 1.

• 1st Excited State: $m_{\pi^0} = 134.97\,\text{MeV}$, $m_{\pi^+} = 139.57\,\text{MeV}$

∴ $\frac{2}{m_{\pi^0}} + \frac{4}{m_{\pi^+}} = 0.0434\,(\text{MeV})^{-1}$; $\frac{5}{m_{\pi^0}} + \frac{1}{m_{\pi^+}} = 0.0442$

↓ 1st excited state

∴ the π^+ goes to the 1st excited state.

∴ $E_{1st\ excited\ state} = \frac{\pi^2\hbar^2}{2L^2}\left(\frac{2}{m_{\pi^0}} + \frac{4}{m_{\pi^+}}\right) = \left(6\frac{\pi^2\hbar^2}{2m_\pi L^2}\ \text{if you set } m_{\pi^0} = m_{\pi^+}\right)$

$\Psi(\underset{\pi^0}{x_1}, \underset{\pi^0}{x_2}, \underset{\pi^+}{x_3}) = \psi_1(x_1)\psi_1(x_2)\psi_2(x_3)$ ← degeneracy = 1

If $m_{\pi^0} = m_{\pi^+}$ then $\Psi(x_1,x_2,x_3) = \psi_1(x_3)\frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1)\right)$ also allowed.

∴ degeneracy = 2.

(c) 5 neutrons & 3 protons: Put 2 neutrons & 2 protons in the ground state. Put 2 neutrons & the remaining proton in the 1st excited state. Put the final neutron in the 2nd excited state: $E_g = \frac{\pi^2\hbar^2}{L^2}\left[\frac{1}{2m_n} + \frac{1}{2m_n} + \frac{1}{2m_p} + \frac{1}{2m_p} + \frac{4}{2m_n} + \frac{4}{2m_n} + \frac{4}{2m_p} + \frac{9}{2m_n}\right]$

$= \frac{\pi^2\hbar^2}{2L^2}\left[\frac{2+8+9}{m_n} + \frac{2+4}{m_p}\right] = \frac{\pi^2\hbar^2}{2L^2}\left[\frac{19}{m_n} + \frac{6}{m_p}\right] = \left(\frac{\pi^2\hbar^2}{2mL^2}(25)\ \text{if } m_n = m_p\right)$

(d) $8\pi^0 + 3\pi^+$. These are all bosons so they can all go in the ground state:

$E_g = \frac{\pi^2\hbar^2}{L^2}\left(\frac{8}{2m_{\pi^0}} + \frac{3}{2m_{\pi^+}}\right) = \left(\frac{\pi^2\hbar^2}{2m_\pi L^2}(11)\ \text{if } m_{\pi^0} = m_{\pi^+}\right)$

Problem 3

$$\Psi = \frac{1}{\sqrt{3!}}\begin{pmatrix} |\alpha\rangle_1 & |\alpha\rangle_2 & |\alpha\rangle_3 \\ |\beta\rangle_1 & |\beta\rangle_2 & |\beta\rangle_3 \\ |\gamma\rangle_1 & |\gamma\rangle_2 & |\gamma\rangle_3 \end{pmatrix} = \frac{1}{\sqrt{3!}}\Big[|\alpha\rangle_1\left(|\beta\rangle_2|\gamma\rangle_3 - |\beta\rangle_3|\gamma\rangle_2\right) - |\alpha\rangle_2\left(|\beta\rangle_1|\gamma\rangle_3 - |\beta\rangle_3|\gamma\rangle_1\right) + |\alpha\rangle_3\left(|\beta\rangle_1|\gamma\rangle_2 - |\beta\rangle_2|\gamma\rangle_1\right)\Big]$$

$$= \frac{1}{\sqrt{3!}}\Big[|\alpha\rangle_1|\beta\rangle_2|\gamma\rangle_3 - |\alpha\rangle_1|\gamma\rangle_2|\beta\rangle_3 - |\beta\rangle_1|\alpha\rangle_2|\gamma\rangle_3 + |\beta\rangle_1|\gamma\rangle_2|\alpha\rangle_3 + |\gamma\rangle_1|\alpha\rangle_2|\beta\rangle_3 - |\gamma\rangle_1|\beta\rangle_2|\alpha\rangle_3\Big]$$

Problem 4

(a) In the ultra relativistic limit, we can ignore mass

$\therefore E = \sqrt{p^2c^2 + m^2c^4} \approx pc = \hbar k c$

$\therefore E_F = \hbar k_F c = \hbar c\,(3\pi^2 \rho)^{1/3}$

Approximating total kinetic energy as NE_F,

$$E_T = E_K - \frac{3}{5}\frac{GM^2}{R} = N\hbar c\,(3\pi^2\rho)^{1/3} - \frac{3}{5}\frac{GM^2}{R}$$

$\rho \simeq \frac{Zd}{AM_p}$ — Z: electrons / nucleus; d: mass density; AM_p: approximate mass of each nucleus

$N \simeq \frac{ZM}{AM_p}$

$d = M \Big/ \frac{4\pi}{3}R^3$

$$= N\hbar c\left(3\pi^2 \frac{Zd}{AM_p}\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R}$$

$$= N\hbar c\left(3\pi^2 \frac{Z}{AM_p}\frac{M}{\pi R^3}\left(\frac{3}{4}\right)\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R}$$

$$= \frac{ZM}{AM_p}\hbar c\left(3\pi^2 \frac{Z}{AM_p}\frac{M}{\pi R^3}\left(\frac{3}{4}\right)\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R}$$

$$= \frac{M^{4/3}}{R}\hbar c\left[\frac{9}{4}\left(\frac{Z}{AM_p}\right)^4 \pi\right]^{1/3} - \frac{3}{5}\frac{GM^2}{R} \Rightarrow \frac{M^{4/3}}{R}\underbrace{\left(\frac{3}{2}\right)^{2/3}\hbar c\left(\frac{Z}{AM_p}\right)^{4/3}\pi^{1/3}}_{b} - \frac{3}{5}\frac{GM^2}{R}$$

For equilibrium $E_T \leq 0$

$$\therefore \frac{M^{4/3}}{R}\left(\frac{3}{2}\right)^{2/3}\hbar c\left(\frac{Z}{AM_p}\right)^{4/3}\pi^{1/3} - \frac{3}{5}\frac{GM^2}{R} \leq 0$$

$$\frac{M^{4/3}}{R}b \leq \frac{3}{5}\frac{GM^2}{R} \to M^{2/3} \geq \frac{5}{3}\frac{b}{G} \to M \geq \left(\frac{5}{3}\frac{b}{G}\right)^{3/2}$$

The critical limit is $E_T = 0 \Rightarrow M_c = \left(\frac{5}{3}\frac{b}{G}\right)^{3/2}$ where b is the same as defined above.

(b) Assuming that everything has been converted to neutrons (and assume $M_n = M_p$):

$N \simeq \frac{M}{M_p}$, $\rho \simeq \frac{d}{M_p}$, $d = \frac{M}{\pi R^3}\frac{3}{4}$

$$\therefore E_T = N\hbar c\left(3\pi^2\frac{d}{M_p}\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R} = \frac{M}{M_p}\hbar c\left(3\pi^2 \frac{1}{M_p}\frac{M}{\pi R^3}\frac{3}{4}\right)^{1/3} - \frac{3}{5}\frac{GM^2}{R}$$

$$= \frac{M^{4/3}}{R}\underbrace{\hbar c\left(\frac{9}{4}\pi\frac{1}{M_p^4}\right)^{1/3}}_{b'} - \frac{3}{5}\frac{GM^2}{R} \quad ; \quad b' = \hbar c\left(\frac{3}{2}\right)^{2/3}\frac{\pi^{1/3}}{M_p^{4/3}} = b\left(\frac{A}{Z}\right)^{3/4}$$

$$\therefore M_c = \left(\frac{5}{3}\frac{b'}{G}\right)^{3/2} = \left(\frac{5}{3G}\left(\frac{A}{Z}\right)^{3/4} b\right)^{3/2} = \left(\frac{A}{Z}\right)^{9/8}\left(\frac{5}{3}\frac{b}{G}\right)^{3/2} = \left(\frac{A}{Z}\right)^{9/8} M_{c\text{ white dwarf}}$$

Problem 5

We are using variational method to compute the ground state of Helium. As a trial wave function, we will use the "screened" ground state hydrogen wavefunctions (i.e. b/c of the second electron cloud, the charge of the nucleus is not Z, but some reduced value λ).

$$\phi(\lambda, r_1, r_2) = \frac{1}{\pi}\left(\frac{\lambda}{a_0}\right)^3 \exp^{-\lambda(r_1+r_2)/a_0}$$

$$\langle H\rangle = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} = \langle\phi|H|\phi\rangle \quad (\phi \text{ is already normalized}).$$

$$\langle\phi|H|\phi\rangle = \int \phi^* H \phi \, d^3\vec{r}_1 \, d^3\vec{r}_2$$

Notice that $H = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{Ze^2}{(4\pi\epsilon_0)r_1} - \frac{Ze^2}{(4\pi\epsilon_0)r_2} + \frac{e^2}{(4\pi\epsilon_0)r_{12}}$

Notice that ϕ is an eigenstate of the first 4 terms if $\lambda = Z$, so rewrite

$$H = \underbrace{-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{e^2}{4\pi\epsilon_0}\left(\frac{\lambda}{r_1} + \frac{\lambda}{r_2}\right)} + \frac{e^2}{4\pi\epsilon_0}\left(\frac{(\lambda - Z)}{r_1} + \frac{(\lambda - Z)}{r_2} + \frac{1}{r_{12}}\right)$$

$$\therefore \langle H\rangle = 2\lambda^2 E_g + (\lambda - Z)\frac{e^2}{4\pi\epsilon_0}\left[\langle\frac{1}{r_1}\rangle + \langle\frac{1}{r_2}\rangle\right] + \frac{e^2}{4\pi\epsilon_0}\langle\frac{1}{r_{12}}\rangle$$

First of all, what is $\langle\frac{1}{r_1}\rangle$ & $\langle\frac{1}{r_2}\rangle$

$$\langle\frac{1}{r_1}\rangle = \int \phi^* \frac{1}{r_1}\phi \, d^3\vec{r}_1 \, d^3\vec{r}_2 = \frac{1}{\pi^2}\left(\frac{\lambda}{a_0}\right)^6 \int \underbrace{e^{-\lambda(r_1+r_2)/a_0}}_{e^{-\lambda r_1/a_0} e^{-\lambda r_2/a_0}} \frac{1}{r_1} \underbrace{e^{-\lambda(r_1+r_2)/a_0}}_{e^{-\lambda r_1/a_0} e^{-\lambda r_2/a_0}} d^3\vec{r}_1 \, d^3\vec{r}_2$$

$$= \underbrace{\left[\frac{1}{\pi}\left(\frac{\lambda}{a_0}\right)^3 \int e^{-\lambda r_1/a_0}\frac{1}{r_1}e^{-\lambda r_1/a_0} d^3\vec{r}_1\right]}_{\shortparallel} \left[\frac{1}{\pi}\left(\frac{\lambda}{a_0}\right)^3 \underbrace{\int e^{-\lambda r_2/a_0} e^{-\lambda r_2/a_0} d^3\vec{r}_2}_{1 \text{ by normalization (This is } \langle\psi_g|\psi_g\rangle \text{ where } \psi_g \text{ is the ground state of hydrogen w/ } Z=\lambda)}\right]$$

$$= \langle\frac{1}{r_1}\rangle_{\text{ground state of hydrogen w/ } Z=\lambda}$$

Recall that for hydrogenic atoms:

$$\left\langle \frac{1}{r} \right\rangle = \frac{Z}{a_0}$$

$$\therefore \quad \left\langle \frac{1}{r_1} \right\rangle = \frac{\lambda}{a_0}$$

A similar argument shows that $\left\langle \frac{1}{r_2} \right\rangle = \frac{\lambda}{a_0}$ as well.

$$\therefore \langle H \rangle = 2\lambda^2 E_g + 2(\lambda - Z)\frac{e^2}{4\pi\epsilon_0}\frac{\lambda}{a_0} + \frac{e^2}{4\pi\epsilon_0}\left\langle \frac{1}{r_{12}} \right\rangle$$

So let us compute $\left\langle \frac{1}{r_{12}} \right\rangle$:

$$\frac{e^2}{4\pi\epsilon_0}\left\langle \frac{1}{r_{12}} \right\rangle = \int |\phi(\lambda, \vec{r}_1, \vec{r}_2)|^2 \frac{e^2}{4\pi\epsilon_0 r_{12}} \, d^3\vec{r}_1 \, d^3\vec{r}_2$$

We see that the only difference b/w $\frac{e^2}{4\pi\epsilon_0}\left\langle \frac{1}{r_{12}} \right\rangle$ & the perturbation calculation done in class is $Z \to \lambda$.

So quoting 10.81, $\frac{e^2}{4\pi\epsilon_0}\left\langle \frac{1}{r_{12}} \right\rangle = \frac{5}{8}\frac{e^2}{(4\pi\epsilon_0)}\frac{\lambda}{a_0}$

$$\therefore \langle H \rangle = 2\lambda^2 E_g + 2(\lambda - Z)\frac{e^2}{4\pi\epsilon_0}\frac{\lambda}{a_0} + \frac{5}{8}\frac{e^2}{4\pi\epsilon_0}\frac{\lambda}{a_0}$$

$$= \Big[2\lambda^2 \underbrace{E_g \frac{4\pi\epsilon_0 a_0}{e^2}}_{=} + 2\lambda^2 - 2\lambda Z + \frac{5}{8}\lambda\Big]\frac{e^2}{4\pi\epsilon_0 a_0}$$

$$\underbrace{-\frac{\alpha^2 m_e c^2}{2}}_{E_g}\underbrace{\frac{\hbar}{\alpha m_e c}}_{a_0}\frac{4\pi\epsilon_0}{e^2} = -\alpha\frac{\hbar c}{2}\frac{4\pi\epsilon_0}{e^2} = -\frac{e^2}{4\pi\epsilon_0 \hbar c}\frac{\hbar c}{2}\frac{4\pi\epsilon_0}{e^2} = -\frac{1}{2}$$

$$= \Big[-\lambda^2 + 2\lambda^2 - 2\lambda Z + \frac{5}{8}\lambda\Big]\frac{e^2}{4\pi\epsilon_0 a_0}$$

$$= \Big[\lambda^2 - 2\lambda Z + \frac{5}{8}\lambda\Big]\frac{e^2}{4\pi\epsilon_0 a_0}$$