Physics 137 B, Spring 2004

Solutions to HW # 8

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Problem 1

$\mathcal{E}(t) = \mathcal{E}_0 \exp\left(-(t/\tau)^2\right) \rightarrow H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q x \mathcal{E}_0 \exp\left(-(t/\tau)^2\right)$

At $t = -\infty$, oscillator in $|0\rangle$.

$$C_b(t) = \frac{1}{\hbar}\int_{-\infty}^{t} dt' \frac{1}{\omega_{b0}(t')}\left(\frac{\partial H(t')}{\partial t'}\right)_{b0} \exp\left[i\int_{-\infty}^{t'} \omega_{b0}(t'')\,dt''\right]$$

- We must find the instantaneous energies:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega x^2 - q x \mathcal{E}_0 \underbrace{\exp\left(-(t/\tau)^2\right)}_{\text{constant at time } = t.}$$

let $x' = x - \frac{q\mathcal{E}_0}{m\omega^2}\exp\left(-(t/\tau)^2\right)$

$$\Rightarrow \frac{1}{2}m\omega^2 x'^2 = \frac{1}{2}m\omega^2\left(x - \frac{q\mathcal{E}_0}{m\omega^2}\exp\left(-(t/\tau)^2\right)\right)^2 = \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}_0\exp\left(-(t/\tau)^2\right) + \frac{1}{2m\omega^2}\left(q\mathcal{E}_0\exp\left(-(t/\tau)^2\right)\right)^2$$

$$\therefore H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x'^2 - \frac{1}{2m\omega^2}\left(q\mathcal{E}_0\exp\left(-(t/\tau)^2\right)\right)^2$$

$$\therefore E_n(t) = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{1}{2m\omega^2}\left(q\mathcal{E}_0\exp\left(-(t/\tau)^2\right)\right)^2$$

- $$\omega_{b0}(t) = \frac{E_b(t) - E_0(t)}{\hbar} = \frac{\hbar\omega\left(b+\frac{1}{2}\right) - \frac{1}{2m\omega^2}\left(q\mathcal{E}_0\exp\left(-(t/\tau)^2\right)\right)^2 - \left(\frac{\hbar\omega}{2} - \frac{1}{2m\omega^2}\left(q\mathcal{E}_0\exp\left(-(t/\tau)^2\right)\right)^2\right)}{\hbar}$$

$= \omega b$

- $\frac{\partial H}{\partial t} = -\frac{2qx\mathcal{E}_0}{\tau^2}\, t\, e^{-(t/\tau)^2}$ (since p & x are independent of time).

$$\left(\frac{\partial H}{\partial t}\right)_{b0} = \langle b|\frac{\partial H}{\partial t}|0\rangle = -\frac{2q\mathcal{E}_0 t}{\tau^2}e^{-(t/\tau)^2}\langle b|x|0\rangle = -\frac{2q\mathcal{E}_0 t}{\tau^2}e^{-(t/\tau)^2}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\delta_{b,1}$$

$\therefore C_{b0} = 0$ unless $b = 1$.

$$\therefore C_{10} = \frac{1}{\hbar}\int_{-\infty}^{t} dt' \frac{1}{\omega}\left(-\frac{2q\mathcal{E}_0 t}{\tau^2}e^{-(t/\tau)^2}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right) e^{i\int_{-\infty}^{t'}\omega\, dt'}$$

$$C_{10} = \frac{1}{\hbar}\int_{-\infty}^{t} dt' \, \frac{1}{\omega}\left(-\frac{2q\mathcal{E}_0 t'}{\tau^2} e^{-(t'/\tau)^2}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\right) e^{i\omega t'}$$

$$= \lim_{t\to\infty}\left[\frac{-2q\mathcal{E}_0}{\hbar\omega\tau^2}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\int_{-\infty}^{t} dt' \, t' \, e^{-\left(\frac{t'}{\tau}\right)^2 + i\omega t'} dt'\right] = -\frac{2q\mathcal{E}_0}{\hbar\omega\tau^2}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\int_{-\infty}^{\infty} dt' \, t' \, e^{-\frac{1}{\tau^2}\left[\left(t' - \frac{1}{2}i\omega\tau^2\right)^2 + \frac{\tau^4}{4}\omega^2\right]}$$

$$= -\frac{2q\mathcal{E}_0}{\tau^2}\left(\frac{1}{2\hbar m\omega^3}\right)^{1/2} e^{-\frac{\tau^2}{4}\omega^2}\int_{-\infty}^{\infty} t' \, e^{-\frac{1}{\tau^2}\left[\left(t' - \frac{1}{2}i\omega\tau^2\right)\right]^2} dt' \qquad \text{let } x = t' - \frac{1}{2}i\omega\tau^2, \; dx = dt'$$

$$= -\frac{2q\mathcal{E}_0}{\tau^2}\left(\frac{1}{\hbar m\omega^3}\right)^{1/2} e^{-\frac{\tau^2}{4}\omega^2}\int_{-\infty}^{\infty}\left(x + \frac{1}{2}i\omega\tau^2\right) e^{-\frac{x^2}{\tau^2}} dx$$

$$= -\frac{2q\mathcal{E}_0}{\tau^2}\left(\frac{1}{\hbar m\omega^3}\right)^{1/2} e^{-\frac{\tau^2}{4}\omega^2}\left[\underbrace{\int_{-\infty}^{\infty} x\, e^{-x^2/\tau^2} dx}_{0 \text{ since odd function}} + \frac{1}{2}i\omega\tau^2 \underbrace{\int_{-\infty}^{\infty} e^{-x^2/\tau^2} dx}_{\sqrt{\pi\tau^2} = \tau\sqrt{\pi}}\right]$$

$$= -\frac{i\omega q\mathcal{E}_0 \tau^{\cancel{3}}}{\cancel{\tau^2}}\left(\frac{\pi}{\hbar m\omega^3}\right)^{1/2} e^{-\tau^2/4\omega^2}$$

$$\text{Prob}_{10} = |C_{10}|^2 = \omega^2 q^2 \mathcal{E}_0^2 \tau^2 \frac{\pi}{\hbar m\omega^3} e^{-\frac{\tau^2}{2}\omega^2} = \frac{(q\mathcal{E}_0\tau)^2}{\hbar m\omega}\pi e^{-\frac{(\tau\omega)^2}{2}}$$

Problem 2

For $t<0$, the system is in the ground state of $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

For $t>0$, $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - q\mathcal{E}_0 x$.

We are asked to show that the only non negligible transition probability to <u>the eigenstates of H_0</u> (not H) is the first excited state. We will follow the discussion starting at the bottom of 461 which continues to 463.

In the weak coupling limit $\frac{|q\mathcal{E}_0|}{(m\hbar\omega^3)^{1/2}} \ll 1$. (Note that this question only makes sense in the weak limit where time dependent pert. theory is also applicable)

Transition to eigenstates of H_0 are given by (9.157):

$$c_b(t) = \sum_n d_n^{(1)} \langle \psi_b^{(0)} | \phi_n^{(1)} \rangle \exp\left[i\left(E_b^{(0)} - E_n^{(1)}\right)t/\hbar\right]$$

$d_n^{(1)}$ is given by 9.165: $d_n^{(1)} = \frac{(-1)^n (\alpha a)^n}{(2^n n!)^{1/2}} \exp\left(-(\alpha a)^2/4\right)$

Since $\frac{|q\mathcal{E}_0|}{(m\hbar\omega^3)^{1/2}} = |\alpha a| \ll 1$, $d_n^{(1)} \approx \frac{(-1)^n}{(2^n n!)^{1/2}}(\alpha a)^n\left(1 - \frac{(\alpha a)^2}{4} + O^4(\alpha a)\right)$

$$\therefore\ d_n^{(1)} \simeq \frac{(-1)^n}{(2^n n!)^{1/2}} (\alpha a)^n$$

We see that as n increases this coefficient becomes negligible:

$$d_0^{(1)} \approx 1$$

$$d_1^{(1)} \simeq -\frac{\alpha a}{2^{1/2}}, \quad d_2^{(2)} \simeq \frac{(\alpha a)^2}{8^{1/2}} \text{ etc.}$$

$\therefore$ let's keep only the 1st two terms in the sum:

$$C_b(t) = d_0^{(1)} \langle \psi_b^{(0)} | \phi_0^{(1)} \rangle e^{i(E_b^{(0)} - E_0^{(1)})t/\hbar} + d_1^{(1)} \langle \psi_b^{(0)} | \phi_1^{(1)} \rangle e^{i(E_b^{(0)} - E_1^{(1)})t/\hbar}$$

$$\langle \psi_b^{(0)} | \phi_0^{(1)} \rangle = \int_{-\infty}^{\infty} \psi_b^*(x)\, \psi_0(x-a)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\alpha}{\sqrt{\pi}\, 2^b b!}\right)^{1/2} e^{-\alpha^2 x^2/2} H_b(\alpha x) \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2 (x-a)^2}{2}} \underbrace{H_0(\alpha(x-a))}_{1}\, dx$$

$$= \frac{\alpha}{\sqrt{\pi}} \left(\frac{1}{2^b b!}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2 + \alpha^2 a x - \frac{\alpha^2 a^2}{2}} H_b(\alpha x)\, dx$$

$$= \frac{\alpha}{\sqrt{\pi}} \left(\frac{1}{2^b b!}\right)^{1/2} e^{-\frac{\alpha^2 a^2}{2}} \int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} H_b(\alpha x)\, dx$$

Consider $\underbrace{\int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} G(\alpha x, t)\, dx}_{\parallel} = \sum_{b=0}^{\infty} \frac{t^b}{b!} \int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} H_b(\alpha x)\, dx$

$$\int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} e^{-t^2 + 2t\alpha x}\, dx = e^{-t^2} \int_{-\infty}^{\infty} e^{-\alpha^2 \left[x^2 - (a + \frac{2t}{\alpha})x\right]}\, dx$$

$$= e^{-t^2} \int_{-\infty}^{\infty} e^{-\alpha^2 \left[\left(x - \frac{1}{2}(a + \frac{2t}{\alpha})\right)^2 - \frac{1}{4}(a + \frac{2t}{\alpha})^2\right]}\, dx$$

$$= e^{-t^2} e^{\alpha^2/4 (a\alpha + 2t)^2} \underbrace{\int_{-\infty}^{\infty} e^{-\alpha^2 \left(x - \frac{1}{2}(a + \frac{2t}{\alpha})\right)^2} dx}_{\sqrt{\pi}/\alpha}$$

$$= \frac{\sqrt{\pi}}{\alpha} e^{-t^2 + \frac{\alpha^2}{4}(a^2\alpha^2 + 4ta\alpha + 4t^2)} = \frac{\sqrt{\pi}}{\alpha} e^{ta\alpha + \frac{1}{4}a^2\alpha^2} = e^{\frac{a^2\alpha^2}{4}} \frac{\sqrt{\pi}}{\alpha} \sum_{b=0}^{\infty} \frac{(a\alpha t)^b}{b!}$$

$$\therefore \sum_{b=0}^{\infty} \left[\frac{\sqrt{\pi}}{\alpha} e^{\frac{a^2\alpha^2}{4}} (a\alpha)^b\right] \frac{t^b}{b!} = \sum_{b=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} H_b(\alpha x)\, dx\right] \frac{t^b}{b!}$$

$$\therefore \int_{-\infty}^{\infty} e^{-\alpha^2 (x^2 - ax)} H_b(\alpha x)\, dx = \frac{\sqrt{\pi}}{\alpha} e^{a^2\alpha^2/4} (a\alpha)^b$$

$$\therefore \langle \psi_b^0 | \phi_0^{(1)} \rangle = \frac{\cancel{\alpha}}{\cancel{\sqrt{\pi}}} \left(\frac{1}{2^b b!} \right)^{1/2} e^{-\alpha^2 a^2/2} \frac{\cancel{\sqrt{\pi}}}{\cancel{\alpha}} e^{a^2\alpha^2/4} (a\alpha)^b = \frac{e^{-\alpha^2 a^2/4}}{(2^b b!)^{1/2}} (a\alpha)^b$$

$$\langle \psi_b^0 | \phi_1^{(1)} \rangle = \int_{-\infty}^{\infty} \psi_b^*(x)\, \psi_1(x-a)\, dx = \left(\frac{\alpha}{\sqrt{\pi}\, 2^b b!} \right)^{1/2} \left(\frac{\alpha}{\sqrt{\pi}\, 2} \right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} H_b(\alpha x)\, e^{-\alpha^2(x-a)^2/2} H_1(\alpha(x-a))$$

$$= \frac{\alpha}{\sqrt{\pi}} \left(\frac{1}{2^{b+1} b!} \right)^{1/2} e^{-\alpha^2 a^2/2} \int_{-\infty}^{\infty} e^{-\alpha^2(x^2 - ax)} H_b(\alpha x)\, 2\alpha(x-a)\, dx$$

$$= \frac{\alpha^2}{\sqrt{\pi}} \left(\frac{1}{2^{b-1} b!} \right)^{1/2} e^{-\alpha^2 a^2/2} \left[\int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} H_b(\alpha x)\, dx - a \underbrace{\int_{-\infty}^{\infty} e^{-\alpha^2(x^2-ax)} H_b(\alpha x)\, dx}_{\frac{\sqrt{\pi}}{\alpha} e^{a^2\alpha^2/4} (a\alpha)^b} \right]$$

Consider $\displaystyle\int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} G(\alpha x, t)\, dx = \sum_{b=0}^{\infty} \frac{t^b}{b!} \int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} H_b(\alpha x)\, dx$

$$L.H.S = \int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} e^{-t^2+2t\alpha x}\, dx = \int_{-\infty}^{\infty} x e^{-\left[(\alpha x - t - a\alpha/2)^2 + a\alpha t - a^2\alpha^2/4\right]}\, dx$$

$$= e^{a\alpha t + \frac{a^2\alpha^2}{4}} \int_{-\infty}^{\infty} x e^{-\left[(\alpha x - t - a\alpha/2)\right]^2}\, dx$$

Let $u = \alpha x - t - a\alpha/2$

$du = \alpha\, dx$

$x = \frac{1}{\alpha}\left[u + t + \frac{a\alpha}{2} \right]$

$$= e^{a\alpha t + \frac{a^2\alpha^2}{4}} \frac{1}{\alpha^2} \underbrace{\int_{-\infty}^{\infty} \left[u + t + \frac{a\alpha}{2} \right] e^{-u^2}\, du}_{\left(t + \frac{a\alpha}{2}\right)\sqrt{\pi}} = \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left(t + \frac{a\alpha}{2}\right) e^{a\alpha t} = \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left(t + \frac{a\alpha}{2}\right) \frac{(a\alpha t)^b}{b!}$$

$$= \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \frac{(a\alpha)^b}{b!} t^{b+1} + \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \frac{(a\alpha)^{b+1}}{2b!} t^b$$

$$= \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left[\frac{(a\alpha)^{b-1}}{(b-1)!} + \frac{(a\alpha)^{b+1}}{2b!} \right] t^b = \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left[\frac{b!\,(a\alpha)^{b-1}}{(b-1)!} + \frac{(a\alpha)^{b+1}}{2} \right] \frac{t^b}{b!}$$

$$= \sum_{b=0}^{\infty} \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left[b(a\alpha)^{b-1} + \frac{(a\alpha)^{b+1}}{2} \right] \frac{t^b}{b!} = \sum_{b=0}^{\infty} \frac{t^b}{b!} \int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} H_b(\alpha x)\, dx$$

$$\therefore \int_{-\infty}^{\infty} x e^{-\alpha^2(x^2-ax)} H_b(\alpha x)\, dx = \frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left[b(a\alpha)^{b-1} + \frac{(a\alpha)^{b+1}}{2} \right]$$

$$\therefore \langle \psi_b^0 | \phi_1^{(1)} \rangle = \frac{\alpha^2}{\sqrt{\pi}} \left(\frac{1}{2^{b-1} b!} \right)^{1/2} e^{-\alpha^2 a^2/2} \left[\frac{\sqrt{\pi}}{\alpha^2} e^{a^2\alpha^2/4} \left(b(a\alpha)^{b-1} + \frac{(a\alpha)^{b+1}}{2} \right) - a\frac{\sqrt{\pi}}{\alpha} e^{a^2\alpha^2/4} (a\alpha)^b \right]$$

$$= \left(\frac{1}{2^{b-1} b!} \right)^{1/2} e^{-\alpha^2 a^2/4} \left[b(a\alpha)^{b-1} + \frac{(a\alpha)^{b+1}}{2} - a\frac{\alpha}{\sqrt{\pi}} (a\alpha)^b \right]$$

$$\therefore c_b(t) = 1 \langle \psi_b^{(0)} | \phi_0^{(1)} \rangle e^{i(E_b^{(0)} - E_0^{(1)})t/\hbar} - \frac{\alpha a}{2^{1/2}} \langle \psi_b^{(0)} | \phi_1^{(1)} \rangle e^{i(E_b^{(0)} - E_1^{(1)})t/\hbar}$$

Again since $a\alpha \ll 1$ we approximate $e^{-\alpha^2 a^2/4} \approx 1$.

$$\therefore c_b(t) \approx \frac{(a\alpha)^b}{(2^b b!)^{1/2}} e^{i(E_b^{(0)} - E_0^{(1)})t/\hbar} - \frac{\alpha a}{2^{1/2}} \left(\frac{1}{2^{b-1} b!} \right)^{1/2} \left[b(a\alpha)^{b-1} + \left(\frac{1}{2} - \frac{1}{\sqrt{\pi}} \right) (a\alpha)^{b+1} \right] e^{i(E_b^{(0)} - E_1^{(1)})t/\hbar}$$

We see that the largest transition prob. is for $b=1$ (transitions to other states are negligible since they are higher than linear order in $a\alpha$).

$$\therefore c_1(t) \approx \frac{(a\alpha)}{2^{1/2}} e^{i(E_1^{(0)} - E_0^{(1)})t/\hbar} - \frac{a\alpha}{2^{1/2}} \left[1 + \left(\frac{1}{2} - \frac{1}{\sqrt{\pi}} \right) \underbrace{(a\alpha)^2}_{O^3(a\alpha)} \right] e^{i(E_1^{(0)} - E_1^{(1)})t/\hbar}$$

$$\approx \frac{a\alpha}{2^{1/2}} \left[e^{i(E_1^{(0)} - E_0^{(1)})t/\hbar} - e^{i(E_1^{(0)} - E_1^{(1)})t/\hbar} \right]$$

$$E_1^{(0)} - E_0^{(1)} = \frac{3}{2}\hbar\omega - \frac{\hbar\omega}{2} + \frac{q^2 \mathcal{E}_0^2}{2m\omega^2} = \hbar\omega + \frac{q^2 \mathcal{E}_0^2}{2m\omega^2} = \hbar\omega \left(1 + \frac{(a\alpha)^2}{2} \right)$$

$$E_1^{(0)} - E_1^{(1)} = \frac{3}{2}\hbar\omega - \frac{3}{2}\hbar\omega + \frac{q^2 \mathcal{E}_0^2}{2m\omega^2} = \hbar\omega \frac{(a\alpha)^2}{2}$$

$$\therefore c_1(t) \approx \frac{a\alpha}{2^{1/2}} \left[e^{i(1 + \frac{(a\alpha)^2}{2})\omega t} - e^{i\frac{(a\alpha)^2}{2}\omega t} \right]$$

Again $e^{i(a\alpha)^2/2\,\omega t} \approx 1$

$$\approx \frac{a\alpha}{2^{1/2}} \left[e^{i\omega t} - 1 \right] = \frac{q\mathcal{E}_0}{(2m\hbar\omega^3)^{1/2}} \left(e^{i\omega t} - 1 \right)$$

$$P_{10}(t) = |c_1(t)|^2 = \frac{q^2 \mathcal{E}_0}{2m\hbar\omega^3} \left(e^{i\omega t} - 1 \right) \left(e^{-i\omega t} - 1 \right)$$

$$= \frac{q^2 \mathcal{E}_0^2}{2m\hbar\omega^3} (2 - 2\cos\omega t) = \frac{4q^2 \mathcal{E}_0^2}{2m\hbar\omega^3} \sin^2\left(\frac{\omega t}{2} \right)$$

$$= \frac{2q^2 \mathcal{E}_0^2}{m\hbar\omega^3} \sin^2\left(\frac{\omega t}{2} \right) \checkmark$$

Problem 3

$$\psi_0^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

at $t=0$ $\quad \psi_n^{(1)} = \sqrt{\frac{2}{2L}} \sin\left(\frac{n\pi x}{2L}\right) = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right)$

Let us be careful with $\psi_0^{(0)} = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) & , 0<x<L \\ 0 & , L<x<2L \end{cases}$

$$\therefore d_n^{(1)}(t) = \int_0^{2L} \psi_0^{(0)}(x)\, \psi_n^{(1)}(x)\, dx = \int_0^{L} \psi_0^{(0)}(x)\, \psi_n^{(1)}(x)\, dx$$

$$= \frac{\sqrt{2}}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{2L}\right) dx = \frac{\sqrt{2}}{L} \int_0^L \left(\frac{e^{\frac{i\pi x}{L}} - e^{-\frac{i\pi x}{L}}}{2i}\right)\left(\frac{e^{\frac{in\pi x}{2L}} - e^{-\frac{in\pi x}{2L}}}{2i}\right) dx$$

$$= -\frac{\sqrt{2}}{4L} \int_0^L e^{i(1+\frac{n}{2})\frac{\pi x}{L}} - e^{i(1-\frac{n}{2})\frac{\pi x}{L}} - e^{-i(1-\frac{n}{2})\frac{\pi x}{L}} + e^{-i(1+\frac{n}{2})\frac{\pi x}{L}}\, dx$$

$$= -\frac{\sqrt{2}}{4L}\left[\frac{L}{i(\frac{n}{2}+1)\pi} e^{i(1+\frac{n}{2})\frac{\pi x}{L}} - \frac{L}{i(1-\frac{n}{2})\pi} e^{i(1-\frac{n}{2})\frac{\pi x}{L}} + \frac{L}{i(1-\frac{n}{2})\pi} e^{-i(1-\frac{n}{2})\frac{\pi x}{L}} - \frac{L}{i(\frac{n}{2}+1)\pi} e^{-i(\frac{n}{2}+1)\frac{\pi x}{L}}\right]_0^L$$

$$= -\frac{\sqrt{2}}{4L}\left[\frac{2L}{(\frac{n}{2}+1)\pi} \sin\left((\frac{n}{2}+1)\frac{\pi x}{L}\right) - \frac{2L}{(1-\frac{n}{2})\pi} \sin\left((1-\frac{n}{2})\frac{\pi x}{L}\right)\right]_0^L$$

$$= -\sqrt{\frac{1}{2}}\left[\frac{1}{(\frac{n}{2}+1)\pi} \underbrace{\sin\left((\frac{n}{2}+1)\pi\right)}_{(-1)^n} - \frac{1}{(1-\frac{n}{2})\pi} \underbrace{\sin\left((1-\frac{n}{2})\pi\right)}_{(-1)^{n+1}}\right] = (-1)^{n+1}\frac{1}{\pi\sqrt{2}}\left[\frac{1}{\frac{n}{2}+1} + \frac{1}{(1-\frac{n}{2})}\right]$$

$$= \frac{(-1)^{n+1}}{\pi\sqrt{2}}\left[\frac{1-\frac{n}{2}+\frac{n}{2}+1}{\frac{n^2}{4}-1}\right] = \frac{(-1)^{n+1}}{\pi\sqrt{2}} \frac{2}{\frac{n^2}{4}-1} = \frac{(-1)^{n+1}}{\pi\sqrt{2}} \frac{8}{n^2-4} =$$

$$P = |d_n^{(1)}|^2 = \frac{8}{\pi^2} \frac{1}{(n^2-4)^2}$$

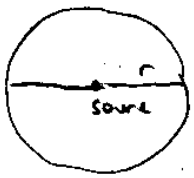
Problem 4

(a) $\lambda = 10m$

In 1 sec 1 J of energy is radiated.

Since the light is monochromatic $E = N\hbar\omega = N\hbar\nu = N\frac{hc}{\lambda}$

$$\Rightarrow N = \frac{\lambda E}{hc} = \frac{10m\ 1J}{6.55\times10^{-34}\ J.s\ \ 3\times10^{8}\ m/s} = 5.09\times10^{25} \text{ photons}$$



$$\# \text{Flux} = \frac{N}{4\pi r^2} = \frac{5.09\times10^{25}}{4\pi\ (10)^2 m^2} = 4.05\times10^{22}\ m^{-2}$$

(b) $\lambda = 10cm$ $\Rightarrow N = \frac{10\times10^{-2}m\ 1J}{6.55\times10^{-34}\ J.s\ \ 3\times10^{8}\ m/s} = 5.09\times10^{22}$ photons

$$\# \text{Flux} = \frac{5.09\times10^{22}}{4\pi\ (10)^2} = 4.05\times10^{19}\ m^{-2}$$

(c) $\lambda = 5890\text{Å} = 5890\times10^{-10}m$ $N = 3\times10^{18}$ photons

$$\# \text{Flux} = 2.39\times10^{15}$$

(d) $\lambda = 1\text{Å} = 1\times10^{-10}m$ $N = 5.1\times10^{14}$ photons

$$\# \text{Flux} = 4.05\times10^{11}$$

Problem 5

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{q}{2m}(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}) + \frac{q^2}{2m}A^2 + q\phi\right]\psi(\vec{r},t) \quad (1)$$

$$\left.\begin{aligned} \vec{A}(\vec{r},t) &= \vec{A}'(\vec{r},t) + \nabla\chi(\vec{r},t) \\ \phi(\vec{r},t) &= \phi'(\vec{r},t) - \frac{\partial}{\partial t}\chi(\vec{r},t) \\ \psi(\vec{r},t) &= \psi'(\vec{r},t)\, e^{iq\chi(\vec{r},t)/\hbar} \end{aligned}\right\} \quad (2)$$

Substitute these eqns to right hand side of (1)

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + i\hbar\frac{q}{2m}\left((A'+\nabla\chi)\cdot\nabla + \nabla\cdot(A'+\nabla\chi)\right) + \frac{q^2}{2m}(A'+\nabla\chi)^2 + q\left(\phi' - \frac{\partial\chi}{\partial t}\right)\right]\psi' e^{\frac{iq\chi}{\hbar}}$$

$$\bullet\ -\frac{\hbar^2}{2m}\nabla^2(\psi' e^{iq\chi/\hbar}) = -\frac{\hbar^2}{2m}\nabla\cdot\left[\nabla(\psi' e^{iq\chi/\hbar})\right] = -\frac{\hbar^2}{2m}\left[\nabla\cdot\left[\nabla\psi' e^{iq\chi/\hbar} + \frac{iq}{\hbar}(\nabla\chi)\psi' e^{iq\chi/\hbar}\right]\right]$$

$$= -\frac{\hbar^2}{2m}\left[\nabla\psi'\cdot\nabla(e^{iq\chi/\hbar}) + \nabla^2\psi' e^{iq\chi/\hbar} + \frac{iq}{\hbar}\left[\nabla\chi\cdot\nabla(\psi' e^{iq\chi/\hbar}) + (\nabla^2\chi)\psi' e^{iq\chi/\hbar}\right]\right]$$

$$= -\frac{\hbar^2}{2m}\left[\frac{iq}{\hbar}\nabla\psi'\cdot\nabla\chi + \nabla^2\psi' + \frac{iq}{\hbar}\nabla\chi\cdot\nabla\psi' - \frac{q^2}{\hbar^2}(\nabla\chi)^2\psi' + \frac{iq}{\hbar}(\nabla^2\chi)\psi'\right] e^{iq\chi/\hbar}$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2\psi' - \frac{i\hbar q}{2m}\left[2\nabla\psi'\cdot\nabla\chi + (\nabla^2\chi)\psi'\right] + \frac{q^2}{2m}(\nabla\chi)^2\psi'\right] e^{iq\chi/\hbar}$$

$$\bullet\ \frac{i\hbar q}{2m}\left[(A'+\nabla\chi)\cdot\nabla + \nabla\cdot(A'+\nabla\chi)\right]\psi' e^{iq\chi/\hbar} = \frac{i\hbar q}{2m}\left[A'\cdot\nabla(\psi' e^{iq\chi/\hbar}) + \nabla\chi\cdot\nabla(\psi' e^{iq\chi/\hbar})\right.$$

$$\left. + \nabla\cdot(A'\psi' e^{iq\chi/\hbar}) + \nabla\cdot(\nabla\chi\psi' e^{iq\chi/\hbar})\right] = \frac{i\hbar q}{2m}\left[(A'\cdot\nabla\psi')e^{iq\chi/\hbar} + \frac{iq}{\hbar}(A'\cdot\nabla\chi)\psi' e^{iq\chi/\hbar} + (\nabla\chi\cdot\nabla\psi')e^{iq\chi/\hbar}\right.$$

$$\left. + \frac{iq}{\hbar}(\nabla\chi)^2\psi' e^{iq\chi/\hbar} + A\cdot\nabla(\psi e^{iq\chi/\hbar}) + (\nabla\cdot A')\psi' e^{iq\chi/\hbar} + \nabla\chi\cdot\nabla(\psi' e^{iq\chi/\hbar}) + (\nabla^2\chi)\psi' e^{iq\chi/\hbar}\right]$$

$$= \frac{i\hbar q}{2m}\left[(A'\cdot\nabla)\psi' + \frac{iq}{\hbar}(A'\cdot\nabla\chi)\psi' + (\nabla\chi\cdot\nabla\psi') + \frac{iq}{\hbar}(\nabla\chi)^2\psi' + (A\cdot\nabla\psi) + \frac{iq}{\hbar}(A\cdot\nabla\chi)\psi' + (\nabla\cdot A')\psi'\right.$$

$$\left. + \nabla\chi\cdot\nabla\psi' + \frac{iq}{\hbar}(\nabla\chi)^2\psi' + (\nabla^2\chi)\psi'\right] e^{iq\chi/\hbar}$$

$$= \frac{i\hbar q}{2m}\left[(A'\cdot\nabla + \nabla\cdot A')\psi' + 2\nabla\chi\cdot\nabla\psi' + (\nabla^2\chi)\psi'\right]e^{iq\chi/\hbar} - \frac{q^2}{2m}\left[2(A'\cdot\nabla\chi)\psi' + 2(\nabla\chi)^2\psi'\right]e^{iq\chi/\hbar}$$

$$\bullet\ \frac{q^2}{2m}\left[(A'+\nabla\chi)\cdot(A'+\nabla\chi)\right]\psi' e^{iq\chi/\hbar} = \frac{q^2}{2m}\left[A'^2\psi' + (A'\cdot\nabla\chi)\psi' + (\nabla\chi\cdot A')\psi' + (\nabla\chi)^2\psi'\right]e^{iq\chi/\hbar}$$

$$= \frac{q^2}{2m}\left[A'^2\psi' + 2(A'\cdot\nabla\chi)\psi' + (\nabla\chi)^2\psi'\right]e^{iq\chi/\hbar}$$

$$\bullet\ q\left(\phi' - \frac{\partial\chi}{\partial t}\right)\psi' e^{iq\chi/\hbar}$$

Putting all of this together:

$$\left[-\frac{\hbar^2}{2m}\nabla^2\psi' - \frac{i\hbar q}{2m}\left[2\cancel{\nabla\psi'\cdot\nabla\chi} + \cancel{(\nabla^2\chi)\psi'}\right] + \cancel{\frac{q^2}{2m}(\nabla\chi)^2\psi'} + \frac{i\hbar q}{2m}\left[(A'\cdot\nabla + \nabla\cdot A')\psi' + 2\cancel{\nabla\psi'\cdot\nabla\chi} + \cancel{(\nabla^2\chi)\psi'}\right]\right.$$

$$\left. - \frac{q^2}{m}\left[\cancel{(A'\cdot\nabla\chi)\psi'} + \cancel{(\nabla\chi)^2\psi'}\right] + \frac{q^2}{2m}A'^2\psi' + \frac{q^2}{m}\cancel{(A'\cdot\nabla\chi)\psi'} + \cancel{\frac{q^2}{2m}(\nabla\chi)^2\psi'} + q\phi'\psi' - q\frac{\partial\chi}{\partial t}\psi'\right]e^{iq\chi/\hbar}$$

$$= \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{2m}(A'\cdot\nabla + \nabla\cdot A') + \frac{q^2}{2m}A'^2 + q\phi'\right]e^{iq\chi/\hbar} - q\frac{\partial\chi}{\partial t}\psi' e^{iq\chi/\hbar}$$

The L.H.S $= i\hbar\frac{\partial}{\partial t}(\psi' e^{iq\chi/\hbar}) = i\hbar\left[\frac{\partial\psi'}{\partial t} + \frac{iq}{\hbar}\frac{\partial\chi}{\partial t}\psi'\right]e^{iq\chi/\hbar} = i\hbar\frac{\partial\psi'}{\partial t}e^{iq\chi/\hbar} - q\frac{\partial\chi}{\partial t}\psi' e^{iq\chi/\hbar}$

$$\Rightarrow i\hbar \frac{\partial \psi'}{\partial t} \cancel{e^{iq\chi/\hbar}} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{2m}(A'\cdot\nabla + \nabla\cdot A') + \frac{q^2}{2m}A'^2 + q\phi'\right]\psi' \cancel{e^{iq\chi/\hbar}}$$

$\therefore$ $\psi = \psi' e^{iq\chi/\hbar}$ satisfies the Schrödinger eqn for $A = A' + \nabla\chi$, $\phi = \phi - \frac{\partial \chi}{\partial t}$

Problem 6

$$i\hbar \frac{\partial}{\partial t}\psi = \left[H_0 - \frac{i\hbar e}{\mu}A\cdot\nabla + \frac{e^2}{2\mu}A^2\right]\psi$$

Let $\psi = \psi' e^{-\frac{i}{\hbar}\frac{e^2}{2\mu}\int^t A^2(t')dt'}$

R.H.S:

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar\left[\frac{\partial\psi'}{\partial t}\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int^t A^2(t')dt'\right) - \frac{i}{\hbar}\frac{e^2}{2\mu}A^2(t)\psi'\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int^t A^2(t')dt'\right)\right]$$

$$= i\hbar\frac{\partial\psi'}{\partial t}\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int^t A^2(t')dt'\right) + \frac{e^2}{2\mu}A^2(t)\psi'\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int^t A^2(t')dt'\right)$$

R.H.S: $\left[H_0 - \frac{i\hbar e}{\mu}A\cdot\nabla + \frac{e^2}{2\mu}A^2\right]\psi'\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int A^2(t')dt'\right)$

$$= \left[H_0 - \frac{i\hbar e}{\mu}A\cdot\nabla\right]\psi'\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int A^2(t')dt'\right) + \frac{e^2}{2\mu}A^2(t)\psi'\exp\left(-\frac{i}{\hbar}\frac{e^2}{2\mu}\int A^2(t')dt'\right)$$

Equate RHS & LHS

$$\Rightarrow i\hbar\frac{\partial\psi'}{\partial t}\cancel{\exp(\ldots)} + \cancel{\frac{e^2}{2\mu}A^2\psi'\exp(\ldots)} = \left[H_0 - \frac{i\hbar e}{\mu}A\cdot\nabla\right]\psi'\cancel{\exp(\ldots)} + \cancel{\frac{e^2}{2\mu}A^2\psi'\exp(\ldots)}$$

$$\Rightarrow i\hbar\frac{\partial\psi'}{\partial t} = \left[H_0 - \frac{i\hbar e}{2\mu}A\cdot\nabla\right]\psi' \quad \checkmark$$