Physics 137 B, Spring 2003

Solutions to PS #9

Arman Cingoz

Problem 1

Half life $t_{1/2}$ is the time when the probability of transition becomes $1/2$.

$P(t) = e^{-t/\tau} \Rightarrow \frac{1}{2} = e^{-t_{1/2}/\tau} \Rightarrow \ln\left(\frac{1}{2}\right) = -t_{1/2}/\tau$

$\Rightarrow t_{1/2} = -\tau \ln\left(\frac{1}{2}\right) = \tau \ln(2)$, where $\tau = \frac{1}{W_{ab}^{s}}$ is the life time.

Problem 2 (Griffiths 9.13)

$|3,0,0\rangle$.

(a) Recall the selection rules, $\Delta \ell = \pm 1$, $\Delta m = \pm 1, 0$

$$|3,0,0\rangle \rightarrow \begin{cases} |2,1,0\rangle \rightarrow |1,0,0\rangle \\ |2,1,1\rangle \rightarrow |1,0,0\rangle \\ |2,1,-1\rangle \rightarrow |1,0,0\rangle \end{cases}$$

(You can't transition from $|300\rangle$ to $|200\rangle$ since $\Delta \ell = 0$.)

(b) We need the transition rate for each route. You only need to compute the transitions from $|3,0,0\rangle \rightarrow |2,1,m_j\rangle$ since we are asked for a decay (not the rate to decay to ground state).

Recall that transition rate $W_{ab}^{s} = \frac{\omega^3 e^2 |\langle \vec{r} \rangle|^2}{3\pi \epsilon_0 \hbar c^3}$ (B.J 11.76 with $|D_{ba}|^2 = e^2 |\langle \vec{r} \rangle|^2$)

since ω is the same for all 3 routes we only need to compute $|\langle \vec{r} \rangle|^2$.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

It takes considerably less time to compute the integrals if you remember some facts from dipole selection rules (11.4 of BJ or pg 316 of Griffiths).
Let me remind you. You have computed long time ago that

$$[L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0.$$

Using the third of these

$$0 = \langle n'l'm' | [L_z, z] | nlm \rangle = \langle n'l'm' | (L_z z - z L_z) | nlm \rangle$$

$$= \langle n'l'm' | (m'\hbar z - z m\hbar) | nlm \rangle = (m'-m)\hbar \langle n'l'm' | z | nlm \rangle$$

$\therefore$ $\boxed{\text{if } m' \neq m, \ \langle n'l'm | z | nlm \rangle = 0 \Rightarrow \text{This comes in handy for routes 2 \& 3}}$

Now use $\langle n'l'm' | [L_z, x] | nlm \rangle = \langle n'l'm' (L_z x - x L_z) | nlm \rangle$

$$= (m'-m)\hbar \langle n'l'm' | x | nlm \rangle = i\hbar \langle n'l'm | y | nlm \rangle$$

$\therefore$ $\boxed{\text{if } m' \neq m, \ (m'-m)\langle n'l'm | x | nlm \rangle = i \langle n'l'm | y | nlm \rangle}$

Now using the last commutation relation you can show that

$\boxed{\text{if } (m'-m)^2 \neq 1 \quad \langle n'l'm' | x | nlm \rangle = \langle n'l'm' | y | nlm \rangle = 0.}$

So let's use these to compute our matrix elements.

<u>Route 1.</u>

$|300\rangle \rightarrow |210\rangle \qquad m' = m \quad \therefore (m'-m)^2 \neq 1 \quad \therefore \langle x \rangle \ \& \ \langle y \rangle = 0$

$\therefore \langle \vec{r} \rangle = \langle z \rangle \Rightarrow \quad \psi_{300} = \frac{1}{\sqrt{4\pi}} \frac{2}{\sqrt{27}} \frac{1}{a^{3/2}} \left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\frac{r^2}{a^2}\right) e^{-r/3a}$

$$\psi_{210} = \sqrt{\frac{3}{4\pi}} \cos\theta \frac{1}{\sqrt{24}} \frac{1}{a^{3/2}} \frac{r}{a} e^{-r/2a}$$

$$\therefore \langle 210 | \vec{r} | 300 \rangle = \langle 210 | z | 300 \rangle = \sqrt{\frac{3}{4\pi}} \frac{1}{\sqrt{24}} \frac{1}{a^{5/2}} \frac{1}{\sqrt{4\pi}} \frac{2}{\sqrt{27}} \frac{1}{a^{3/2}} \int \cos\theta \, r e^{-r/2a} \cdot$$

$$\left(1 - \frac{2}{3a} r + \frac{2}{27a^2} r^2\right) e^{-r/3a} \, r\cos\theta \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{(1/a^4)}{12\sqrt{6}\,\pi} \int_0^\infty r^4 \left(1 - \frac{2}{3a} r + \frac{2}{27a^2} r^2\right) e^{-5r/6a} dr \int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \cdots = \frac{(1/a^4)}{12\sqrt{6}\pi}\left(\frac{4!\,6^5a^5}{5^6}\right)\left(\frac{2}{3}\right)(2\pi) = \frac{2^8 3^4}{5^6\sqrt{6}}a \quad ; \omega_1 \sim \left|\frac{2^8 3^4}{5^6\sqrt{6}}a\right|^2 = \frac{2^{15}3^7}{5^{12}}a^2$$

<u>Route 2 & 3</u>

$|300\rangle \to |21\pm1\rangle \quad m' \neq m \quad \therefore \langle z\rangle = 0$ & $\pm\langle 21\pm1|x|300\rangle = i\langle 21\pm1|y|300\rangle$

So let's compute $\langle x\rangle$: $\quad \psi_{21\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}\frac{1}{\sqrt{24}}\frac{1}{a^{3/2}}\frac{r}{a}e^{-r/2a}$

$$\langle 21\pm1|x|300\rangle = \mp\sqrt{\frac{3}{8\pi}}\frac{1}{\sqrt{24}}\frac{1}{a^{5/2}}\frac{1}{\sqrt{4\pi}}\frac{2}{\sqrt{27}}\frac{1}{a^{3/2}}\int \sin\theta\, e^{\pm i\phi}\, r e^{-r/2a}\left(1-\frac{2}{3}\frac{r}{a}+\frac{2}{27}\frac{r^2}{a^2}\right)e^{-r/3a}$$

$$\cdot\, r\sin\theta\cos\phi\,\sin\theta\; r^2\,dr\,d\theta\,d\phi$$

$$= \mp\frac{1}{24\sqrt{3}\pi a^4}\int_0^\infty r^4\left(1-\frac{2}{3a}r+\frac{2}{27a^2}r^2\right)e^{-5r/6a}dr\int_0^\pi \sin^3\theta\,d\theta\int_0^{2\pi}(\cos\phi\pm i\sin\phi)\cos\phi\,d\phi$$

$$= \mp\frac{1}{24\sqrt{3}\pi a^4}\left\{4!\left(\frac{6a}{5}\right)^5 - \frac{2}{3a}5!\left(\frac{6a}{5}\right)^6 + \frac{2}{27a^2}6!\left(\frac{6a}{5}\right)^7\right\}\left(\frac{4}{3}\right)(\pi)$$

$$= \mp\frac{1}{18\sqrt{3}a^4}4!\frac{6^5a^5}{5^7}\left(25 - \frac{2}{3a}\cdot 5\cdot 5\cdot 6a + \frac{2}{27a^2}\cdot 6\cdot 5\cdot 6^2a^2\right) = \mp\frac{4a}{3\sqrt{3}}\cdot\frac{6^5}{5^7}(25-100+80)$$

$$= \mp\frac{2^7 3^4}{5^6\sqrt{3}}a$$

$$\therefore |\langle r\rangle|^2 = \left[\mp\frac{2^7 3^4}{5^6\sqrt{3}}a \pm i\frac{2^7 3^4}{5^6\sqrt{3}}a\right]^2 = \frac{2^{14}3^8a^2}{5^{12}\,3} + \frac{2^{14}3^8a^2}{5^{12}\,3} = \frac{2^{15}3^7}{5^{12}}a^2$$

$\therefore |\langle 210|\vec{r}|300\rangle|^2 = |\langle 21\pm1|\vec{r}|300\rangle|^2 \quad \therefore$ all the rates are equal.

$\therefore$ The fraction for each route is $1/3$

(c) So the life time is 3 times the 1 over the transition rate of any decay route.

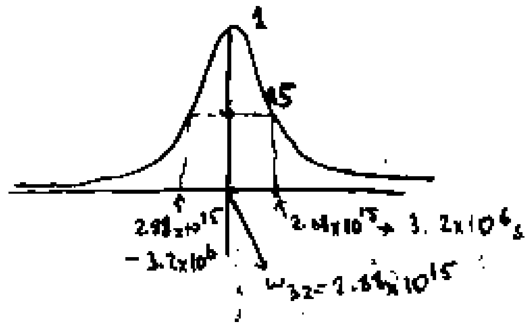
$$\text{rate} = R = \frac{3\omega^3 e^2|\langle\vec{r}\rangle|^2}{3\pi\epsilon_0\hbar c^3}, \quad \omega = \frac{E_3-E_2}{\hbar} = \frac{1}{\hbar}\left(\frac{E_1}{9}-\frac{E_1}{4}\right) = -\frac{5}{36}\frac{E_1}{\hbar}$$

$$= \frac{e^2}{\pi\epsilon_0\hbar c^3}\left(-\frac{5}{36}\frac{E_1}{\hbar}\right)^3\left(\frac{2^{15}3^7}{5^{12}}a^2\right) = 6\left(\frac{2}{5}\right)^9\left(\frac{E_1}{mc^2}\right)^2\left(\frac{c}{a}\right)$$

$= 6\left(\frac{2}{3}\right)^9 \left(\frac{13.6}{.511\times10^6}\right)^2\left(\frac{3\times10^8}{0.529\times10^{-10}}\right)\frac{1}{sec} = 6.32\times10^6\ \frac{1}{s}$

$\therefore\ \tau = \frac{1}{R} = 1.58\times10^{-7}\,s$

(d) $\Gamma = \frac{\hbar}{\tau} \Rightarrow \frac{\Gamma}{2\hbar} = \frac{\hbar}{\tau}\frac{1}{2\hbar} = \frac{1}{2\tau}$; $\omega_{32} = -\frac{5}{36}\frac{E_1}{\hbar} = -\frac{5}{36}\left(\frac{-13.6\times1.6\times10^{-19}\,J}{1.05\times10^{-34}\,J\cdot s}\right) = 2.88\times10^{15}\,1/s$



Problem 3 (Griffiths 6.19)

In the weak field limit, the Zeeman interaction is a perturbation to the fine structure. The total energies are given by

$$E = E_{nj} + E_Z = \frac{-13.6\,eV}{n^2}\left[1+\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right] + \mu_B\, g_J\, m_j\, B_{ext}\ ;\ g_J = 1+\frac{j(j+1)-\ell(\ell+1)+3/4}{2j(j+1)}$$

let's write down our 8 states: $s=\frac{1}{2}$, $m_s = \pm\frac{1}{2}$ always

$|n\ \ell\ j\ m_j\rangle$

1 $|2\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$
2 $|2\,0\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$ $\Big\}$ $g_J = \left[1+\frac{1/2(3/2)+3/4}{2(1/2)(3/2)}\right] = 1+\frac{3/2}{3/2} = 2$

3 $|2\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$
4 $|2\,1\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$ $\Big\}$ $g_J = \left[1+\frac{1/2(3/2)-1(2)+3/4}{2(1/2)(3/2)}\right] = 1-\frac{1/2}{3/2} = \frac{2}{3}$

(states 1–4) $E_{nj} = -\frac{13.6}{4}\left[1+\frac{\alpha^2}{4}\left(\frac{2}{1}-\frac{3}{4}\right)\right] = -3.4\,eV\left(1+\frac{5}{16}\alpha^2\right)$

5 $|2\,1\,\tfrac{3}{2}\,\tfrac{3}{2}\rangle$
6 $|2\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle$
7 $|2\,1\,\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$
8 $|2\,1\,\tfrac{3}{2}\,-\tfrac{3}{2}\rangle$ $\Big\}$ $g_J = \left[1+\frac{(3/2)(5/2)-1(2)+3/4}{2(3/2)(5/2)}\right] = 1+\frac{5/2}{15/2} = \frac{4}{3}$ $\Big\}$ $E_{nj} = -3.4\,eV\left[1+\frac{\alpha^2}{4}\left(\frac{2}{2}-\frac{3}{4}\right)\right] = -3.4\,eV\left(1+\frac{\alpha^2}{16}\right)$

$\therefore$ The energies are:

$E_1 = -3.4\,eV\left(1+\frac{5}{16}\alpha^2\right) + \mu_B B_{ext}$	$E_5 = -3.4\,eV\left(1+\frac{1}{16}\alpha^2\right) + 2\mu_B B_{ext}$
$E_2 =$ " $-\mu_B B_{ext}$	$E_6 =$ " $+\frac{2}{3}\mu_B B_{ext}$
$E_3 =$ " $+\frac{1}{3}\mu_B B_{ext}$	$E_7 =$ " $-\frac{2}{3}\mu_B B_{ext}$
$E_4 =$ " $-\frac{1}{3}\mu_B B_{ext}$	$E_8 =$ " $-2\mu_B B_{ext}$

Problem 4

In the strong field limit, the fine structure is the perturbation. Total energy is

$$E = \underbrace{\overbrace{-\frac{13.6\,eV}{n^2}}^{=-3.4\,eV} + \mu_B B_{ext}(m_\ell + 2m_s)}_{\text{unperturbed piece}} + \overbrace{\frac{13.6\,eV}{n^3}}^{=1.7\,eV \text{ for } n=2}\alpha^2\left\{\frac{3}{4n} - \left[\frac{\ell(\ell+1) - m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right]\right\}$$

STATE $\lvert n\,\ell\,m_\ell\,m_s\rangle$	$(m_\ell + 2m_s)$	$\frac{3}{4n} - \left(\frac{\ell(\ell+1)-m_\ell m_s}{\ell(\ell+1/2)(\ell+1)}\right)$	TOTAL ENERGIES
1 $\lvert 2\,0\,0\,1/2\rangle$	1	$-5/8$	$-3.4\,eV\left(1+\frac{5}{16}\alpha^2\right) + \mu_B B_{ext}$
2 $\lvert 2\,0\,0\,-1/2\rangle$	-1	$-5/8$	" $-\mu_B B_{ext}$
3 $\lvert 2\,1\,1\,1/2\rangle$	2	$-1/8$	$-3.4\,eV\left(1+\frac{1}{16}\alpha^2\right) + 2\mu_B B_{ext}$
4 $\lvert 2\,1\,-1\,-1/2\rangle$	-2	$-1/8$	" $-2\mu_B B_{ext}$
5 $\lvert 2\,1\,0\,1/2\rangle$	1	$-7/24$	$-3.4\,eV\left(1+\frac{7}{48}\alpha^2\right) + \mu_B B_{ext}$
6 $\lvert 2\,1\,0\,-1/2\rangle$	-1	$-7/24$	" " $-\mu_B B_{ext}$
7 $\lvert 2\,1\,1\,-1/2\rangle$	0	$-11/24$	$-3.4\,eV\left(1+\frac{11}{48}\alpha^2\right)$
8 $\lvert 2\,1\,-1\,1/2\rangle$	0	$-11/24$	$-3.4\,eV\left(1+\frac{11}{48}\alpha^2\right)$

If you ignore the fine structure contribution as B & J does then you get:

$\left.\begin{matrix}\lvert 2\,0\,0\,1/2\rangle \\ \lvert 2\,1\,0\,1/2\rangle\end{matrix}\right\}$ $-3.4\,eV + \mu_B B_{ext}$

$\left.\begin{matrix}\lvert 2\,0\,0\,-1/2\rangle \\ \lvert 2\,1\,0\,-1/2\rangle\end{matrix}\right\}$ $-3.4\,eV - \mu_B B_{ext}$

$\lvert 2\,1\,1\,1/2\rangle$ $-3.4 + 2\mu_B B_{ext}$

$\lvert 2\,1\,-1\,-1/2\rangle$ $-3.4 - 2\mu_B B_{ext}$

$\left.\begin{matrix}\lvert 2\,1\,1\,-1/2\rangle \\ \lvert 2\,1\,-1\,1/2\rangle\end{matrix}\right\}$ $-3.4\,eV$