SPIN:

- Intrinsic angular momentum associated with some particles. Fermions are particles that have half-integer spin. Bosons are particles that have integer spin.
- Spin satisfy the same commutation relations as orbital angular momentum.

$$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k \quad , [S^2, S_i] = 0 \quad \text{where } S^2 = S_x^2 + S_y^2 + S_z^2 = \vec{S}\cdot\vec{S}$$

$$S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$$

$$S_z|s,m\rangle = m\hbar|s,m\rangle \quad (-s \le m \le s)$$

(You could also choose $|s,m\rangle$ to be eigenvectors of S^2 & S_x or S^2 & S_y as well)

$$S_\pm|s,m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s, m\pm 1\rangle$$

- SPIN 1/2: Electrons, quarks, protons, neutrons all have spin 1/2.

  There are two basis vectors $|s=\frac{1}{2}, m=\frac{1}{2}\rangle$, $|s=\frac{1}{2}, m=-\frac{1}{2}\rangle$

  In the basis where $|1/2, 1/2\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$ & $|1/2, -1/2\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

Remember that $S_x = \frac{1}{2}(S_+ + S_-) \rightarrow S_x|1/2, 1/2\rangle = \frac{\hbar}{2}|1/2, -\frac{1}{2}\rangle$, $S_x|\frac{1}{2}, -\frac{1}{2}\rangle = \frac{\hbar}{2}|1/2, 1/2\rangle$

$S_y = \frac{1}{2i}(S_+ - S_-) \rightarrow S_y|1/2, 1/2\rangle = -\frac{\hbar}{2i}|\frac{1}{2}, -\frac{1}{2}\rangle$, $S_y|\frac{1}{2}, -\frac{1}{2}\rangle = \frac{\hbar}{2i}|\frac{1}{2}, \frac{1}{2}\rangle$

- READ & UNDERSTAND pg 158 & 159 of Griffiths, if you have access to it.
- SPIN in a Magnetic Field:

  Magnetic dipole moment $\vec{\mu}$ is proportional to spin with the proportionality constant γ, gyromagnetic ratio. $\rightarrow \vec{\mu} = \gamma\vec{S}$

  Recall the classical E&M eqns: W (Energy) $= -\vec{\mu}\cdot\vec{B}$

  N (Torque) $= \vec{\mu}\times\vec{B}$

  F (Force) $= -\nabla W = \nabla(\vec{\mu}\cdot\vec{B})$

  These equations are identical in QM if you replace angular momentum in $\vec{\mu}$ with the spin angular momentum operator.

$\therefore\ \hat{H} = -\vec{\mu}\cdot\vec{B}$ ($\hat{H}$ is the Hamiltonian, i.e. energy operator).

- **ADDITION OF ANGULAR MOMENTA (ORBITAL OR SPIN):**

  - You could add 2 particles with spins s_1 & s_2, or you could add the orbital angular momentum of 1 particle to its spin angular momentum

$$\Rightarrow \vec{S}_{TOT} = \vec{S}_1 + \vec{S}_2 \quad \text{or} \quad \vec{J} = \vec{L} + \vec{S}$$

The rest of the total angular momentum operators are defined as (I will use J, L, S notation; for 2 spins just replace $S_{TOT} \leftrightarrow J$, $S_1 \leftrightarrow L$, $S_2 \leftrightarrow S$)

$$\vec{J} = \vec{L} + \vec{S}\ ;\ J_z = L_z + S_z,\ J_x = L_x + S_x,\ J_y = L_y + S_y\ ;\ J_+ = L_+ + S_+,\ J_- = L_- + S_-$$

$$J^2 = (\vec{J}\cdot\vec{J}) = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) = L^2 + S^2 + 2\vec{L}\cdot\vec{S} = L^2 + S^2 + 2(L_xS_x + L_yS_y + L_zS_z)$$

- There are 2 ways to write the total angular momentum eigenstates.

  (1) $|j, m_j\rangle$ such that $J^2|j,m_j\rangle = \hbar^2 j(j+1)|j,m_j\rangle$
  $J_z|j,m_j\rangle = m_j\hbar\,|j,m_j\rangle$

  (2) $$|j,m_j\rangle = \sum_{\substack{m_\ell\ m_s \\ (\text{s.t } m_\ell + m_s = m_j)}} C^{\ell s j}_{m_\ell m_s m_j}\ |\ell, m_\ell\rangle \otimes |s, m_s\rangle$$

  $C^{\ell s j}_{m_\ell m_s m_j}$ are called the Clebsh-Gordon Coefficients.

  You could also invert this equation:

  $$|\ell, m_\ell\rangle|s, m_s\rangle = \sum_j C^{\ell s j}_{m_\ell m_s m_j}\,|j\ m_j\rangle$$

- Let's discuss some of the rules that must be observed when adding 2 angular momenta.

  (1) Mathematically all we are doing is combining 2 vector spaces into one larger vector space (denoted by the operation $\otimes$). The simplest basis for this new vector space is the simple product of the basis vectors of the two original spaces:

$$\underset{\substack{\downarrow \\ \text{dimension} \\ n}}{\ell} \otimes \underset{\substack{\downarrow \\ \text{dimension} \\ m}}{s} = \underset{\substack{\downarrow \\ \text{dimension} \\ n\times m}}{j}$$

For example: let $\ell = 2$, $s = 1/2$

The basis vectors for ℓ are $|2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle, |2,-2\rangle$ = 5 dimensional vector space.

The basis vectors for s are $|1/2, 1/2\rangle, |1/2, -1/2\rangle$ = 2 dim. vector space.

$\therefore$ The basis vectors for j are: $|2,2\rangle|1/2,1/2\rangle$; $|2,2\rangle|1/2,-1/2\rangle$; $|2,1\rangle|1/2,1/2\rangle$; $|2,1\rangle|1/2,-1/2\rangle$

$|2,0\rangle|1/2,1/2\rangle$; $|2,0\rangle|1/2,-1/2\rangle$; $|2,-1\rangle|1/2,1/2\rangle$; $|2,-1\rangle|1/2,-1/2\rangle$

$|2,-2\rangle|1/2,1/2\rangle$; $|2,-2\rangle|1/2,-1/2\rangle$

There are 10 of them (i.e 2×5).

(2) Now, we have a basis for the total angular momentum. As it turns out this is not a very nice basis because the basis vectors we have written down do not generally correspond to one j. We must take linear combinations of them to get definite j states (this is where the $C^{\ell s j}_{m_\ell m_s m_j}$ come in).

I will state 2 rules w/o proof. The proof is a little tedious. If you are interested check out the second volume of Cohen-Tannoudji's "Quantum Mechanics" book (an excellent reference for formalism as well, although the notation is a little strange).

(a) For given ℓ & s, j ranges from $\ell+s$ to $|\ell-s|$ <u>in integer steps</u> ie $\ell+s, \ell+s-1, \ell+s-2, \dots |\ell-s|$

(b) For a given $|j,m_j\rangle$, the $|\ell,m_\ell\rangle\otimes|s,m_s\rangle$ states that go into the linear combination must satisfy $m_\ell + m_s = m_j$. (This one isn't too hard to see since $J_z = L_z + S_z$)

Mathematically, one writes $\ell \otimes s = j_1 \oplus j_2 \oplus \dots$ etc.

For example: $\ell=2$, $s=1/2$. The allowed j values are $2+1/2 = 5/2$ to $2-1/2 = 3/2$

$\therefore$ $2 \otimes 1/2 = \frac{5}{2} \oplus \frac{3}{2}$

The explanation of what $\oplus$ means requires Group Theory, but you can simply think of breaking up the large vector into sub vector spaces where the basis vectors for each subspace are the eigenstates of a specific J^2 operator (corresponding to j_1 or j_2, etc).

Let's see this in an example. On the last page, we saw that for $\ell = 2$ & $s = 1/2$, we constructed a 10 dimensional vector space. We also saw that $2 \otimes 1/2$ could be decomposed into $j = \frac{5}{2}$ & $j = \frac{3}{2}$. Let's see if we still get a 10 dimensional vector space:

for $j = \frac{5}{2}$, we have the basis vectors: $|\frac{5}{2}, \frac{5}{2}\rangle, |\frac{5}{2}, \frac{3}{2}\rangle, |\frac{5}{2}, \frac{1}{2}\rangle, |\frac{5}{2}, -\frac{1}{2}\rangle$
$|\frac{5}{2}, -\frac{3}{2}\rangle, |\frac{5}{2}, -\frac{5}{2}\rangle \rightarrow$ 6 dimensional space.

$j = \frac{3}{2}$, we have the basis vectors: $|\frac{3}{2}, \frac{3}{2}\rangle, |\frac{3}{2}, \frac{1}{2}\rangle, |\frac{3}{2}, -\frac{1}{2}\rangle, |\frac{3}{2}, -\frac{3}{2}\rangle$
↳ 4 dimensional space

Notice that $6 + 4 = 10$. We have decomposed the 10 dimensional space into 2 subvector spaces w/ dim 6 & dim 4.

(3) Now, the final question is how the basis vectors of these subspaces are related to the naive basis we wrote down for the whole space on the last page. THIS IS WHERE YOU USE THE CLEBSCH-GORDON COEFFICIENTS:

I will discuss both how to derive these coefficients & how to read them off of a Clebsch-Gordon Coeff. Table (Note: unfortunately Brandsen does not have a CG Table. The prof. has linked a website where you can download one, or look at Griffiths pg 168).

(3a) <u>How to derive Clebsch-Gordon Coefficients:</u>

To illustrate the method, I'll use a specific example. Let $\ell = 1$ & $s = \frac{1}{2}$. Therefore, we want to construct the vector space $1 \otimes \frac{1}{2}$.

First, let's write down the naive basis:

Basis vectors for $l=1$: $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$. $\leftarrow$ dim = 3

Basis vectors for $s=\frac{1}{2}$: $|\frac{1}{2},\frac{1}{2}\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle$ $\leftarrow$ dim = 2

Basis vectors for $1\otimes\frac{1}{2}$: $|\underset{l}{1},\underset{m_l}{1}\rangle|\underset{s}{\frac{1}{2}},\underset{m_s}{\frac{1}{2}}\rangle$, $|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle$, $|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle$, $|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle$

$|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle$, $|1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle$

As expected we have $3\otimes 2=6$ dimensional vector space.

Now, we use the first rule given in section (2): j ranges from $l+s$ to $|l-s|$ in integer steps. $\therefore$ the allowed values of j are:

$$j=1+\frac{1}{2}=\frac{3}{2} \quad \text{to} \quad j=1-\frac{1}{2}=\frac{1}{2}$$

Since there is no integer b/w $\frac{3}{2}$ & $\frac{1}{2}$: $1\otimes\frac{1}{2}=\frac{3}{2}\oplus\frac{1}{2}$

The basis vectors for $j=\frac{3}{2}$ ($|j,m_j\rangle$): $|\frac{3}{2},\frac{3}{2}\rangle$, $|\frac{3}{2}\ \frac{1}{2}\rangle$, $|\frac{3}{2}\ -\frac{1}{2}\rangle$, $|\frac{3}{2},-\frac{3}{2}\rangle$

The basis vectors for $j=\frac{1}{2}$: $|\frac{1}{2},\frac{1}{2}\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle$

Now, what we want to do is to write these vectors in terms of the basis we came up with at the top of this page. This is where the second rule given in section (2) comes in handy: $m_j=m_l+m_s$:

Finally, here is the procedure: First, look at the highest j states ($\frac{3}{2}$ in this case)

(1) Notice that the highest m_j state allowed (in this case $|\frac{3}{2},\frac{3}{2}\rangle$) has to be equal the highest m_l times m_s state (in this case $|1,1\rangle|\frac{1}{2},\frac{1}{2}\rangle$)
The reason for this is simple: Only way to get $m_j=\frac{3}{2}$ is $m_l=1+m_s=\frac{1}{2}=\frac{3}{2}$
There is no other combination of m_l & m_s that will give us $\frac{3}{2}$

$$\therefore\ |\tfrac{3}{2},\tfrac{3}{2}\rangle=|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

(2) Similarly, the lowest m_j state ($|\frac{3}{2},-\frac{3}{2}\rangle$) can be written as

$|\frac{3}{2},-\frac{3}{2}\rangle=|1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle$ (The reason again is that the only way $m_j=-\frac{3}{2}$ is if $m_l=-1$ & $m_s=-\frac{1}{2}$)

So, now we have to find the states in between (i.e $|\frac{3}{2}\frac{1}{2}\rangle, |\frac{3}{2},\frac{-1}{2}\rangle$)

There is a systematic way of finding these but before I introduce that let's use the 2^{nd} rule again to see which states can contribute to $|\frac{3}{2},\frac{1}{2}\rangle$ & $|\frac{3}{2},-\frac{1}{2}\rangle$

• $|\frac{3}{2},\frac{1}{2}\rangle$: Since $m_j=\frac{1}{2}$, there are 2 ways:
$m_\ell = 1,\ m_s=-\frac{1}{2}$ or $m_\ell=0,\ m_s=\frac{1}{2}$

$$\therefore |\tfrac{3}{2},\tfrac{1}{2}\rangle = a\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + b\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

• $|\frac{3}{2},-\frac{1}{2}\rangle$: Since $m_j=-\frac{1}{2}$, there are 2 ways:
$m_\ell=0,\ m_s=-\frac{1}{2}$; or $m_\ell=-1,\ m_s=\frac{1}{2}$

$$\therefore |\tfrac{3}{2},-\tfrac{1}{2}\rangle = c\,|1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + d\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Here is the systematic way to find a, b, c, and d: Use raising & lowering operators.

$$J_-|\tfrac{3}{2},\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\;|\tfrac{3}{2},\tfrac{1}{2}\rangle = \hbar\sqrt{3}\;|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$\therefore |\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{1}{\hbar\sqrt{3}}J_-|\tfrac{3}{2},\tfrac{3}{2}\rangle = \frac{1}{\hbar\sqrt{3}}(L_-+S_-)\,|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

(**NOTE**: L operators do not affect $|s,m_s\rangle$ vectors. & S operators do not affect $|\ell,m_\ell\rangle$ vectors.)

$$=\frac{1}{\hbar\sqrt{3}}\Big[L_-\big(|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + S_-\,|1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle\Big] = \frac{1}{\hbar\sqrt{3}}\Big[\hbar\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$+\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big] = \frac{1}{\hbar\sqrt{3}}\Big[\hbar\sqrt{2}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \hbar\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle\Big]$$

$$=\sqrt{\tfrac{2}{3}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \qquad \therefore\ a=\sqrt{\tfrac{1}{3}},\ b=\sqrt{\tfrac{2}{3}}$$

Now, to find $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$, you can either act on $\left|\frac{3}{2},\frac{1}{2}\right\rangle$ with the lowering operator or you can act on $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ with the raising operator. The second method is usually easier to use. So,

$$J_+\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)+\frac{3}{2}\left(\frac{-3}{2}+1\right)}\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \hbar\sqrt{3}\left|\frac{3}{2},-\frac{1}{2}\right\rangle$$

$$\therefore \left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\hbar\sqrt{3}}J_+\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \frac{1}{\hbar\sqrt{3}}\left(L_+ + S_+\right)|1,-1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$= \frac{1}{\hbar\sqrt{3}}\left[L_+|1,-1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + S_+|1,-1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right] = \frac{1}{\hbar\sqrt{3}}\left[\hbar\sqrt{1(1+1)+1(-1+1)}\,|1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right.$$

$$\left. + \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)+\frac{1}{2}\left(-\frac{1}{2}+1\right)}\,|1,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right] = \frac{1}{\hbar\sqrt{3}}\left[\hbar\sqrt{2}\,|1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \hbar\,|1,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle\right]$$

$$= \sqrt{\frac{2}{3}}\,|1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle \quad \therefore\ c=\sqrt{\frac{2}{3}},\ d=\sqrt{\frac{1}{3}}$$

So Let's summarize:

$$\left|\frac{3}{2}\,\frac{3}{2}\right\rangle = |1,1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2},+\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,|1,0\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1,1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,|1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1,-1\rangle\left|\frac{1}{2},+\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2},\frac{-3}{2}\right\rangle = |1,-1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Now we have to do the same thing for $j=\frac{1}{2}$ states:

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = e\,|1,0\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle + f\,|1,1\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

(The reason this is the form of the linear combination is that $m_j=\frac{1}{2}$ $\therefore m_\ell=0, m_s=\frac{1}{2}$ or $m_\ell=1,\ m_s=-\frac{1}{2}$).

Now, recall that $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ has to be orthogonal to $\left|\frac{3}{2},\frac{1}{2}\right\rangle$ since they are both basis vectors in our 6 dimensional vector space. So, all we need to do is to guess e & f s.t. $\left\langle\frac{1}{2},\frac{1}{2}\middle|\frac{3}{2},\frac{1}{2}\right\rangle=0$

$$\left\langle \tfrac{1}{2},\tfrac{1}{2} \,\middle|\, \tfrac{3}{2},\tfrac{1}{2} \right\rangle = \left(e^*\langle 1,0|\langle \tfrac{1}{2},\tfrac{1}{2}| + f^*\langle 1,1|\langle \tfrac{1}{2},-\tfrac{1}{2}|\right)\left(\sqrt{\tfrac{2}{3}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

(Assume $e^* = e$, $f^* = f$: this is just a phase convention).

$\therefore\ e\sqrt{\frac{2}{3}} + f\sqrt{\frac{1}{3}} = 0$ (All other terms are zero by orthonormality)

so if I choose $e = -\sqrt{\frac{1}{3}}$ & $f = +\sqrt{\frac{2}{3}}$, then this equation is satisfied.

$$\therefore\ |\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

A similar argument reveals that $|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle$

(If you don't want to guess, just lower $|\frac{1}{2},\frac{1}{2}\rangle$ using J_-)

3(b) Clebsch-Gordon Table:

I hope you are still with me because this is where it gets easier. Obviously, this process is very time consuming and should be repeated until one has understood the procedure. Once you have understood the procedure, you should just use the table that tabulates all these coefficients... unfortunately, it takes a little getting used to. CG Tables consist of bunch of interlocking blocks. On the top left corner of each block is $n \times m$ where n & m are integer or half integers. These refer to l & s or s_1 & s_2. Below, I will copy the $1 \times \frac{1}{2}$ block to illustrate how to use it.

$l \to 1 \times \frac{1}{2} \leftarrow s_2$

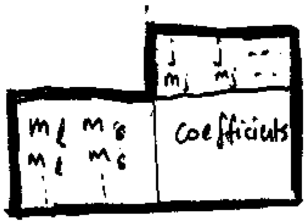
m_l	m_s	$j = 3/2$, $m_j = +3/2$
+1	+1/2	1

m_l	m_s	$j = 3/2$, $m_j = +1/2$	$j = 1/2$, $m_j = +1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_l	m_s	$j = 3/2$, $m_j = -1/2$	$j = 1/2$, $m_j = -1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_l	m_s	$j = 3/2$, $m_j = -3/2$
−1	−1/2	1

Each $n \times m$ block on the table is divided into sub blocks by thick lines. In each, subblock there is another sub block that has 2 edges that are thin lines & 2 edges that are thick lines. This is the coefficients block:



- In the coefficients block, it is understood that every entry must be square rooted. For entries that are negative, it is understood that the negative sign is outside the square root.

- To write $|j, m_j\rangle$ in terms of $|l, m_l\rangle|s, m_s\rangle$, you find the j, m_j column and read down the coefficients list. Each entry is the coefficient multiplying the $|l m_l\rangle|s, m_s\rangle$ state listed to the left.

- You can also invert this procedure and write $|l, m_l\rangle|s, m_s\rangle$ in terms of $|j, m_j\rangle$. To do this, find the row with $|l, m_l\rangle|s, m_s\rangle$ you want and read across the row for the coefficients. Each entry is the coefficient multiplying the $|j, m_j\rangle$ state listed on the top.

FINAL REMARKS:

- Although I called the $|l, m_l\rangle|s, m_s\rangle$ basis a naive basis, it has its uses. (Someone might give you a spin 1/2 particle with known total angular momentum j and ask for measurements of L)

- If you were paying close attention on the derivation of the $|\frac{1}{2}, \frac{1}{2}\rangle$ state, you might have realized that $e = \sqrt{\frac{1}{3}}$, $f = -\sqrt{\frac{2}{3}}$ would also work. (Also, e & f could be imaginary instead of real). All of these are phase conventions that do not alter physical quantities which depend on $\langle\psi|\psi\rangle$