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The Standard Electroweak Gauge Model

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1 QED

Gauge invariance is a central feature of field theories. It ensures calculated observables are finite, that is, the amplitudes in a perturbation expansion are renormalizable. QED was the first gauge theory and the simplest.

The quantum-mechanical description of experiments on charged particles could be made invariant under local phase transformations on the particle wavefunction if we introduced a long-range field (thus, massless) coupled to charge that is the electromagnetic field, and if we made a suitable simultaneous local gauge transformations on the e.m. potential.

We can reproduce this argument in mathmaticl form. The free field Lagrangian of a spin 1/2 particle is

$$L = \bar{\psi}(i\gamma^{\mu}\delta_m u - m)\psi \tag{1}$$

Consider a phase rotation on the charged-particle wavefunction $\psi(x)$, where x is a space-time variable.

$$\psi(x) \to \psi'(x) = e^{ie\theta(x)}\psi(x) \tag{2}$$

where $\theta(x)$ is arbitrary. Equations of motion have a term $\psi^*(x)\delta_\mu\psi(x)$, and we would like this term to be invariant under the local phase rotation. Let's calcuate this term under the phase rotation:

$$\begin{aligned}
\psi(x)^*\delta_\mu\psi(x) \to \psi'(x)^* \cdot \delta_\mu\psi'(x)^* &= e^{-ie\theta(x)}\psi(x) \cdot e^{ie\theta(x)}[\delta_\mu + ie\delta_\mu\theta(x)]\psi(x) \\
&= \psi(x)^*\delta_\mu\psi(x) + ie[\delta_\mu\theta(x)]\psi(x)^*\psi(x) \\
&\neq \psi(x)^*\delta_\mu\psi(x) \qquad (3)
\end{aligned}$$

Thus this term is not invariant. However, if we introduce potential $A_\mu(x)$ of a field which transforms

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \delta_\mu\theta(x) \tag{4}$$

and the derivative δ_μ is replaced with the covariant derivative

$$D_\mu = \delta_\mu - ieA_\mu(x), \; ^{1} \tag{5}$$

$\psi(x)^* D_\mu \psi(x)$ will be invariant:

$$\begin{aligned}
\psi(x)^* D_\mu \psi(x) &= \psi(x)^*[\delta_\mu - ieA_\mu(x)]\psi(x) \qquad (6)\\
\rightarrow \psi'(x)^* \cdot [\delta_\mu - ieA'_\mu(x)]\psi'(x) &= e^{-ie\theta(x)}\psi(x)^* \cdot [\delta_\mu - ie(A_\mu(x) + \delta_\mu\theta(x))]e^{ie\theta(x)}\psi(x)\\
&= e^{-ie\theta(x)}\psi(x)^* \cdot [\delta_\mu + ie\delta_\mu\theta(x) - ieA_\mu(x) - ie\delta_\mu\theta(x)]\psi(x)\\
&= \psi(x)^*[\delta_\mu - ieA_\mu(x)]\psi(x)\\
&= \psi(x)^* D_\mu \psi(x)
\end{aligned}$$

The Lagrangian term $\bar{\psi}(x)i\gamma^\mu\delta_\mu\psi(x)$ will change to $\bar{\psi}(x)i\gamma^\mu D_\mu\psi(x)$, and the kinetic term of $A_\mu(x)$ needs to be added to the Lagrangian. Thus the local gauge invariant Lagrangian will be written as

$$L = -\frac{1}{4}F^{\mu\nu}_{\mu\nu} + \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \qquad (7)$$

where $F_{\mu\nu}$ is the e.m. field tensor

$$F_{\mu\nu} = \delta_\mu A_\nu - \delta_\nu A_\mu. \qquad (8)$$

$F_{\mu\nu}F^{\mu\nu}$ is invariant under local gauge transformations, and the Lagrangian is invariant under local gauge transformations:

$$\begin{aligned}
\psi(x) &\rightarrow \psi'(x) = e^{ie\theta(x)}\psi(x) = e^{ieQ\theta(x)}\psi(x) \qquad (9)\\
A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \delta_\mu\theta(x) \qquad (10)
\end{aligned}$$

where Q is the charge operator: $Q\psi(x) = -\psi(x)$ if $\psi(x)$ is the electron field. D_μ specifies the form of the interaction (eA_μ) of the charged particle.

In summary, local gauge invariance demands that there be a gauge field A_μ which interacts with fermions in a prespribed way. Since fermion mass term and gauge boson mass term ($m_A A_\mu A^\mu$) are not invariant under local gauge transformations, both fermions and gauge bosons are massless.[2]

Let's reconsider the interaction term in the Lagrangian, $e[\bar{\psi}(x)\gamma^\mu\psi(x)]A_\mu = eJ^\mu_{em}A_\mu$ where $J^\mu_{em} = \bar{\psi}(x)\gamma^\mu\psi(x)$ is the electromagnetic current. When we use Euler-Lagrange eq.

$$\delta_\mu[\delta L/\delta(\delta_\mu\phi(x))] = \delta L/\delta\phi(x) \qquad (11)$$

[2]Gauge invariance fails if the fermions and bosons are massive and can only be restored by including extra interactions (Higgs scalars).

obtained from the requirement of stationary action for any field $\phi(x)$, with $\phi(x) = A_\nu(x)$.

$$\begin{aligned} left\ side &= \delta_\mu[\delta L/\delta(\delta_\mu A_\nu)] = -\delta_\mu F^{\mu\nu} \\ right\ side &= \delta L/\delta A_n u = -eJ^\nu_{em} \end{aligned}$$

$$\delta_\mu F^{\mu\nu} = eJ^\nu_{em}. \tag{12}$$

These are equations of motion for e.m.

$$\begin{aligned} \delta_\nu \delta_\mu F^{\mu\nu} &= e\delta_\nu J^\nu_{em} \\ 0 &= e\delta_\nu J^\nu_{em} \end{aligned} \tag{13}$$

since

$$\begin{aligned} \delta_\nu \delta_\mu F^{\mu\nu} &= \delta_\nu \delta_\mu (\delta^\mu A^\nu - \delta^\nu A^\mu) \\ &= \delta_\nu \delta_\mu \delta^\mu A^\nu - \delta_\nu \delta_\mu \delta^\nu A^\mu = \delta_\mu \delta_\nu \delta^\mu A^\nu - \delta_\nu \delta_\mu \delta^\nu A^\mu = \delta_\nu \delta_\mu \delta^\nu A^\mu - \delta_\nu \delta_\mu \delta^\nu A^\mu = 0. \end{aligned}$$

This represents the charge conservation, Noether's theorem.

In the language of group theory, the Gauge transformations with scalar phase $\theta(x)$ belong to a unitary group $U(x) = e^{ie\theta(x)} = e^{ieQ\theta(x)}$. $U(1)$ is formed by the infinite set of phase transformations. The full Lagrangian is said to have the symmetry $U^Q(1)$ with the charge operator Q as the generator. Since $\theta(x)$ is a scalar quantity, $U(1)$ group is Abelian.

2 Generalized Gauge Invariance, SU(2)

More complex phase transformations are possible, specified by non commuting operators, forming non-Abelian groups. Such gauge transformations were proposed by Yang and Mill in 1954, and involved fields containing both charged and neutral massless bosons. Specifically, they chose the group SU(2) of isospin, which involves the non-commuting pauli matrices, $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$. Recall the conservation of isospin in strong interactions implies invariance under an isospin rotation.

$$\psi \to \psi' = e^{ig\vec{\tau}\cdot\vec{\Lambda}}\psi \tag{14}$$

where g is constant, and $\vec{\Lambda}(x)$ is an arbitrary vector (similar to $\theta(x)$ in U(1)) which can be rotated in isospin space. $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ is an isospin operator (similar to Q in U(1)) and τ^i $(i = 1, 2, 3)$ are the generators of SU(2) symmetry transformations. The infinite set of $e^{ig\vec{\tau}\cdot\vec{\Lambda}(x)}$ forms a SU(2) group.

In the same way as for the electromagnetic interactions, a gauge invariant description can be obtained by introducing a massless isovector field $\vec{W}_\mu$ with a strong coupling g (analogous to e in U(1)). The covariant derivative is

$$D_\mu = \delta_\mu - ig\vec{\tau} \cdot \vec{W}_\mu \tag{15}$$

where $\vec{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$ (1,2,3 are three components in isospin space, and μ denotes space-time components), and

$$\vec{W}_\mu \to \vec{W}'_\mu = \vec{W}_\mu + \delta_\mu \vec{\Lambda} - g\vec{\Lambda} \times \vec{W}_\mu \tag{16}$$

where the third term has to be included because isospin matrices $\vec{\tau}$ do not commute. Let's prove $\psi^* D_\mu \psi$ is invariant under SU(2) gauge transformations by considering Λ is infinitesimal (infinitesimal is sufficient since finite transformations can be made from them by integrations), thus $e^{ig\vec{\tau}\cdot\vec{\Lambda}} \simeq 1 + ig\vec{\tau} \cdot \vec{\Lambda}$:

$$\begin{aligned}
\psi^* D_\mu \psi &= \psi^*(\delta_\mu - ig\vec{\tau} \cdot \vec{W}_\mu)\psi \\
\to \psi'^* D_\mu \psi' &= \psi^*(1 - ig\vec{\tau} \cdot \vec{\Lambda})[\delta_\mu - ig\vec{\tau} \cdot (\vec{W}_\mu + \delta_\mu \vec{\Lambda} - g\vec{\Lambda} \times \vec{W}_\mu)](1 + ig\vec{\tau} \cdot \vec{\Lambda})\psi \\
&= \psi^*(\delta_\mu - ig\vec{\tau} \cdot \vec{W}_\mu)\psi \\
&+ g^2\psi^*[-(\vec{\tau} \cdot \vec{\Lambda})(\vec{\tau} \cdot \vec{W}_\mu) + (\vec{\tau} \cdot \vec{W}_\mu)(\vec{\tau} \cdot \vec{\Lambda}) + i\vec{\tau} \cdot (\vec{\Lambda} \times \vec{W}_\mu)]\psi \\
&+ O(\Lambda^2)
\end{aligned}$$

Including the kinetic term of $\vec{W}_\mu$ field, the Lagrangian is

$$\begin{aligned}
L &= L_{gauge} + L_{fermi} \\
&= -\frac{1}{4}\vec{W}_{\mu\nu} \cdot \vec{W}^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.
\end{aligned} \tag{17}$$

and

$$\vec{W}_{\mu\nu} = \delta_\mu \vec{W}_\nu - \delta_\nu \vec{W}_\mu - g\vec{W}_\mu \times \vec{W}_\nu \tag{18}$$

where the first two terms represent kinetic energy and the third term $-g\vec{W}_\mu \times \vec{W}_\nu$ represents the self-coupling of $\vec{W}_\mu$ fields which is essential for gauge invariance. One can show that $\vec{W}_{\mu\nu} \cdot \vec{W}^{\mu\nu}$ is gauge invariant, but $M^2\vec{W}_\mu \cdot \vec{W}^\mu$ is not. Thus $\vec{W}$ fields must be massless. The interaction term L_{int} is then

$$\begin{aligned}
L_{int} &= g(\bar{\psi}\gamma^\mu\vec{\tau}\psi) \cdot \vec{W}_\mu \\
&= g\vec{J}_\mu \cdot \vec{W}_\mu
\end{aligned} \tag{19}$$

where $\vec{J}_\mu = \psi\gamma^\mu\vec{\tau}W$ is the strong isospin current.

SU(2) gauge model is a candidate for weak interactions where $\vec{W}$ can represent three weak gauge bosons $W^+, W^- W^0$, where

$$\begin{aligned} W_\mu^+ &= \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ W_\mu^- &= \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2) \\ W_\mu^+ &= W_\mu^3 \end{aligned} \tag{20}$$

For weak interactions, left-handed structure of charged current couplings (only left handed lepton and right handed anti-lepton can be coupled to W) needs to be implemented. There has been desire to unify weak interactions and electromagnetic interactions in a single gauge theory.

3 The Weinberg-Salam SU(2) $\times$ U(1) Model

In 1967 - 1968, Weinberg and Salam proposed a gauge theory unifying weak and electromagnetic interactions, "electroweak interactions", based on an SU(2) group of weak isospin I and a $\mathrm{U}^Y(1)$ group of weak hypercharge Y, where $Y = Q - I_3$ and I_3 is the third component of weak isospin. Vector bosons from this model are $\vec{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$ for a SU(2) massless isovector triplet and B_μ for a U(1) massless isovector singlet.

3.1 Lagrangian

The SU(2) $\times$ $\mathrm{U}^Y(1)$ Lagrangian is

$$\begin{aligned} L &= L_{gauge} + L_{fermi} \\ &= -\frac{1}{4}\vec{W}_{\mu\nu}\cdot\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \bar{\psi}i\gamma^\mu D_\mu\psi \end{aligned} \tag{21}$$

$$L_{gauge} = -\frac{1}{4}\vec{W}_{\mu\nu}\cdot\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{22}$$

$$L_{fermi} = \bar{\psi}i\gamma^\mu D_\mu\psi \tag{23}$$

where

$$\vec{W}_{\mu\mu} = \delta_\mu\vec{W}_\nu - \delta_\nu\vec{W}_\mu - g\vec{W}_\mu\times\vec{W}_\nu \tag{24}$$

$$B_{\mu\nu} = \delta_\mu B_\nu - \delta_\nu B_\mu \tag{25}$$

$$D_\mu = \delta_\mu - ig\vec{\tau}\cdot\vec{W}_\mu - ig'YB_\mu \tag{26}$$

Field functions and gauge bosons are transformed in the following way;

$$\psi \rightarrow \psi' = e^{ig\vec{\tau}\cdot\vec{\Lambda}(x)}e^{ig'Y\beta(x)}\psi \tag{27}$$

$$\vec{W}_\mu \rightarrow \vec{W}'_\mu = \vec{W}_\mu + \delta_\mu\Lambda - g\vec{\Lambda}\times\vec{W} \tag{28}$$

$$B_\mu \rightarrow B'_\mu = B_\mu + \delta_\mu\beta \tag{29}$$

The interaction term is then

$$\begin{aligned} L_{int} &= g(\bar{\psi}\gamma^\mu\vec{\tau}\psi)\vec{W}_\mu + g'(\bar{\psi}\gamma^\mu Y\psi)B_\mu \\ &= g\vec{J}_\mu\cdot\vec{W}_\mu + g'J^Y_\mu B_\mu \end{aligned} \tag{30}$$

where $\vec{J}_\mu$ and J^Y_μ are weak isospin and hypercharge currents and g and g' are couplings to $\vec{W}_\mu$ and B_μ, respectively. $Y = Q - I_3$, and hypercharge current

$$J^Y_\mu = J^{e.m.}_\mu - J^3_\mu \tag{31}$$

contains both the electroweak current and isospin third component current. Thus one can create A_μ (em field) and Z_μ (neutral weak field) from a linear combination of B_μ and W^3_μ.

$$\begin{aligned} B_\mu &= \cos\theta_W A_\mu - \sin\theta_W Z_\mu \\ W^3_\mu &= \sin\theta_W A_\mu + \cos\theta_W Z_\mu \end{aligned}$$

where θ_W, called "mixing angle" or "Weinberg angle", which is not predicted by this model and needs be measured from experiments. Using definition

$$\frac{g'}{g} = \tan\theta_W, \tag{32}$$

$$B_\mu = \frac{-g'Z_\mu + gA_\mu}{\sqrt{q^2 + q'^2}} \tag{33}$$

$$W^3_\mu = \frac{gZ_\mu + g'A_\mu}{\sqrt{q^2 + q'^2}} \tag{34}$$

Let's rewrite the Largrangian interaction term using A_μ, Z_μ, $J^+_\mu = J^1_\mu + iJ^-_\mu$, and $J^-_\mu = J^1_\mu - iJ^-_\mu$.

$$\begin{aligned} L_{int} &= g(\vec{J}_\mu\cdot\vec{W}_\mu) + g'(J^Y_\mu B_\mu) \\ &= g[J^1_\mu W^1_\mu + J^2_\mu W^2_\mu + J^3_\mu W^3_\mu] + g'[J^{em}_\mu - J^3_\mu]B_\mu \\ &= \frac{1}{\sqrt{2}}g[J^1_\mu(W^+_\mu + W^-_\mu) + J^2_\mu i(W^+_\mu - W^-_\mu) + [gJ^3_\mu\cos\theta_W - g'(J^{em}_\mu - J^3_\mu)\sin\theta_W]Z_\mu \end{aligned} \tag{35}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}} g[(J_\mu^1 + iJ_\mu^2)W_\mu^+ (J_\mu^1 - iJ_\mu^2)W_\mu^- \\
&+ \frac{g}{\cos\theta_W}[\cos^2\theta_W J_\mu^3 + \sin^2\theta_W J_\mu^{em} + \sin^2\theta_W J_\mu^3] + g\sin\theta_W [J_\mu^3 + J_\mu^{em} - J_\mu^3] Z_\mu \\
&= \frac{g}{\sqrt{2}}[J_\mu^+ W_\mu^+ + J_\mu^- W_\mu^-]A_\mu + \frac{g}{\cos\theta_W}[J_\mu^3 - \sin^2\theta_W J_\mu^{em}]Z_\mu + g\sin\theta_W J_\mu^{em}
\end{aligned}
$$

where the first, second, and third term represents weak charged current, weak neutral current and electromagnetic neutral current, respectively. Since the couple in the third term is the elementary charge e,

$$
e = g\ \sin\theta_W. \tag{36}
$$

Also let's define

$$
g_Z = \frac{g}{\cos\theta_W} = \frac{e}{\sin\theta_W \cos\theta_W}. \tag{37}
$$

Thus

$$
\begin{aligned}
L_{int} &= \frac{g}{\sqrt{2}}[J_\mu^+ W_\mu^+ + J_\mu^- W_\mu^-]A_\mu + g_Z[J_\mu^3 - \sin^2\theta_W J_\mu^{em}]Z_\mu + eJ_\mu^{em} && (38) \\
D_\mu &= \delta_\mu - ig\vec{\tau}\cdot\vec{W}_\mu - ig' B_\mu \\
&= \delta_\mu - ieQA_\mu - \frac{ig}{\sqrt{2}}(\tau^+ W_\mu^+ + \tau^- W_\mu^-) - ig_Z(\tau^3 - \sin^2\theta_W\ Q)Z_\mu && (39)
\end{aligned}
$$

where $\tau^+ = \tau^1 + i\tau^2$ and $\tau^- = \tau^1 - i\tau^2$. Let's rewrite L_{fermi} :

$$
\begin{aligned}
L_{fermi} &= \bar{\psi} i\gamma^\mu \delta_\mu \psi && (40) \\
&+ e\bar{\psi}\gamma^\mu (Q\psi) A_\mu \\
&+ \frac{g}{\sqrt{2}}\bar{\psi}\gamma^\mu(\tau^+\psi)W_\mu^+ + \frac{g}{\sqrt{2}}\bar{\psi}\gamma^\mu(\tau^-\psi)W_\mu^- \\
&+ g_Z\ \bar{\psi}\gamma^\mu(\tau^3\psi)Z_\mu - g_Z\ \sin^2\theta_W\ \bar{\psi}\gamma^\mu(Q\psi)Z_\mu
\end{aligned}
$$

The Lagrangian is invariant under the infinitesimal local gauge transformations, SL(2)$_L$ and U(1)$_Y$ independently. Here is how fermion and gauge boson fields are transformed:

SU(2)$_L$	U(1)$_Y$
$\psi_L \to [1 + ig\vec{\tau}\cdot\vec{\Lambda}(x)]\ \psi_L$	$\psi_L \to [1 + ig' Y\beta(x)]\ \psi_L$
$\psi_R \to \psi_R$	$\psi_R \to [1 + ig' Y\beta(x)]\ \psi_R$
$\vec{W}_\mu \to \vec{W}_\mu + \delta_\mu\vec{\Lambda}(x) - g\vec{\Lambda}(x)\times\vec{W}_\mu$	$\vec{W}_\mu \to \vec{W}_\mu$
$B_\mu \to B_\mu$	$B_\mu \to B_\mu + \delta_\mu\beta(x)$

3.2 Labeling the quark and lepton states

- lepton SU(2) doublet $L = (\nu_L, e_L)$:

  one state can be changed to another by rotation in electroweak SU(2) space (analogy to rotation in strong isospin space between a neutron and a proton). In electroweak SU(2) space, W bosons cannect the members of an electroweak doublet.

- lepton SU(2) singlet e_R

- quark SU(2) doublet $Q_L = (u_L, d_L)$

- quark SU(1) single u_R and d_R.

Their weak isospin, electric quark and hypercharge quantum numbers are

	I	I_3	Q	Y
ν_L	$\frac{1}{2}$	$+\frac{1}{2}$	0	$-\frac{1}{2}$
e_L	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$
u_L	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{2}{3}$	$\frac{1}{6}$
d_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
e_R	0	0	-1	-1
u_R	0	0	$+\frac{2}{3}$	$+\frac{2}{3}$
d_R	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$

3.3 Largangian with quark/lepton lebeling

The first generation Lagrangian is

$$\begin{aligned}
L_{fermi} &= \Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{f} i\gamma^\mu D_\mu f \qquad (41)\\
&= \Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{f} i\gamma^\mu \delta_\mu f \\
&+ e\Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{\gamma}^\mu (Qf) A_\mu \\
&+ \frac{g}{\sqrt{s}} \Sigma_{(f=L,Q_L)} \bar{f}\gamma^\mu(\tau^+ f) W_\mu^+ + \frac{g}{\sqrt{s}} \Sigma_{(f=L,Q_L)} \bar{f}\gamma^\mu(\tau^- f) W_\mu^- \\
&+ g_Z \Sigma_{(f=L,Q_L)} \bar{f}\gamma^\mu(\tau^3 f) Z_\mu - g_Z \sin^2\theta_W \Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{f}\gamma^\mu (Qf) Z_\mu \\
&= \Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{f} i\gamma^\mu \delta_\mu f \\
&+ e\Sigma_{(f=L,e_R,Q_L,u_R,d_R)} \bar{\gamma}^\mu (Qf) A_\mu \\
&+ \frac{g}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu \bar{e}_L + \bar{u}_L \gamma^\mu d_L) W_\mu^+ + \frac{g}{\sqrt{2}} (\bar{e}_L \gamma^\mu \bar{\nu}_L + \bar{d}_L \gamma^\mu u_L) W_\mu^-
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{g_Z}{2}(\nu_L\gamma^\mu\nu_L - \bar{e}_L\gamma^\mu e_L + u_L\gamma^\mu u_L - d_L\gamma^\mu d_L)Z_\mu \\
&+ \quad g_Z\ \sin^2\theta_W[\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R - \frac{2}{3}\bar{u}_L\gamma^\mu u_L + \frac{1}{3}\bar{d}_L\gamma^\mu d_L + \frac{2}{3}\bar{u}_R\gamma^\mu u_R - \frac{1}{3}\bar{d}_R\gamma^\mu d_R]Z_\mu
\end{aligned}
$$

The fermion couplings to $W^\pm$ bosons are the same for all the fermions, $g/\sqrt{2}$. However the couplings to Z bosons are different. Let's rewrite terms with Z_μ couplings.

$$
\begin{aligned}
L_{\bar{f}fZ} &= g_Z[\frac{1}{2}]\bar{\nu}_L\gamma^\mu\nu_L \qquad (42) \\
&+ g_Z[-\frac{1}{2} + \sin^2\theta_W]\bar{e}_L\gamma^\mu e_L Z_\mu \\
&+ g_Z[+\sin^2\theta_W]\bar{e}_R\gamma^\mu e_R Z_\mu \\
&+ g_Z[+\frac{1}{2} - \frac{2}{3}\sin^2\theta_W]\bar{u}_L\gamma^\mu u_L Z_\mu \\
&+ g_Z[-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W]\bar{u}_L\gamma^\mu u_L Z_\mu \\
&+ g_Z[+\frac{2}{3}\sin^2\theta_W]\bar{u}_R\gamma^\mu u_R Z_\mu \\
&+ g_Z[-\frac{1}{3}\sin^2\theta_W]\bar{d}_R\gamma^\mu d_R Z_\mu \\
&= \frac{g}{\cos\theta_W\sin\theta_W}\Sigma_{(f=L,e_R,Q_L,u_R,d_R}\ [I_3 - Q\sin^2\theta_W]\bar{f}\gamma^\mu f Z_\mu \\
&= \frac{g}{\cos\theta_W\sin\theta_W}[\Sigma_{(f=L,Q_L)}\ g_L\bar{f}\gamma^\mu f Z_\mu + \Sigma_{(f=e_R,u_R,d_R)}\ g_R\bar{f}\gamma^\mu f Z_\mu] \qquad (43) \\
coupling &= [I_3 - Q\sin^2\theta_W]\frac{g}{\cos\theta_W\sin\theta_W} \qquad (44) \\
g_L &= I_3 - Q\sin^2\theta_W \qquad (45) \\
g_R &= -Q\sin^2\theta_W \qquad (46)
\end{aligned}
$$

$I_3 - Q\sin^2\theta_W$ values are summarized the table below.

fermion	$I_3 - Q\sin^2\theta_W$
ν_L	$+\frac{1}{2}$
e_L	$-\frac{1}{2} + \sin^2\theta_W$
e_R	$\sin^2\theta_W$
u_L	$+\frac{1}{2} - \frac{2}{3}\sin^2\theta_W$
d_L	$-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W$
u_R	$-\frac{2}{3}\sin^2\theta_W$
d_R	$+\frac{1}{3}\sin^2\theta_W$

Often so-called "vector coupling (c_V)" and "axial-vector coupling (c_A)" are used. They are defined to be

$$
\begin{aligned}
c_v &= g_L + g_R = I_3 - 2Q\sin^2\theta_W && (47)\\
c_A &= g_L - g_R = I_3 && (48)
\end{aligned}
$$

since vector (axial-vector) interactions couple right-handed and left-handed states with the same (opposite) sign. Sometimes c_V and c_A include $\frac{1}{\sin\theta_W\cos\theta_W}$ which comes from the θ_W factor in g_Z:

$$
\begin{aligned}
c_v &= \frac{I_3 - 2Q\sin^2\theta_W}{\sin\theta_W\cos\theta_W} && (49)\\
c_A &= \frac{I_3}{\sin\theta_W\cos\theta_W} && (50)
\end{aligned}
$$

fermion	g_L	g_R	$c_V = g_L + g_R$ $= I_3 - 2Q\sin^2\theta_W$	$c_A = g_L - g_R$ $= I_3$
ν	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$
e	$-\frac{1}{2} + \sin^2\theta_W$	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$	
u	$+\frac{1}{2} - \frac{2}{3}\sin^2\theta_W$	$-\frac{2}{3}\sin^2\theta_W$	$+\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$+\frac{1}{2}$
d	$-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W$	$+\frac{1}{3}\sin^2\theta_W$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$

A few remarks:

- Let's take a quick look at the fermion-W^+ interaction term,

$$
\begin{aligned}
\bar{\nu}_L\gamma^\mu e_L &= \frac{1}{2}\nu_L\gamma^\mu(1-\gamma^5)e && (51)\\
&= \frac{1}{2}\nu_L\gamma^\mu e - \frac{1}{2}\nu_L\gamma^\mu\gamma^5 e
\end{aligned}
$$

  where the first term transforms like a normal vector and the second term like an axial-vector, thus (V - A) charged current interaction.

- Helicity of a fermion is unchanged by a vector or axial-vector interaction if the fermion is massless

$$
\begin{aligned}
\bar{f}_L\gamma^\mu f_R &= \bar{f}\frac{1}{2}(1-\gamma^5)\gamma^\mu\frac{1}{2}(1+\gamma^5)f = \frac{1}{4}\bar{f}\gamma^\mu(1-\gamma^5)f = 0\\
\bar{f}_L\gamma^\mu\gamma^5 f_R &= 0
\end{aligned}
$$

- Strength of weak interactions

– Let's pick $(\bar{\nu}_L \gamma^\mu e_L) W_\mu^+$ term.

– From the Lagrangian, weak charge is $g/\sqrt{2}$

$$\begin{aligned} \alpha_{weak} &= \frac{(g/\sqrt{2})^2}{4\pi} \\ &= \frac{(g^2/2)}{4\pi} \\ &= \frac{e^2}{4\pi} \frac{1}{2\ \sin^2\theta_W} \\ &\simeq \frac{1}{137} \frac{2}{4\ \sin^2\theta_W} \\ &\simeq \frac{2}{137} \end{aligned} \tag{52}$$

– Rather than being somewhat larger than electromagnetic interactions, the known "weak" interactions were several orders of magnitude smaller because of the W mass in the propergator.

4 The Higgs mechanism

In the Standard Model (SM), a SU(2) doublet of scalar fields (Higgs fields) introduced in order to implement gauge invariant fermion and boson mass terms.

$$L_\phi = (D_\mu \phi)^2 - V(\phi^2) \tag{53}$$

where $\phi^2 = \phi\phi$ and $phi = (\phi^+, \phi^0)$. ϕ^+ and ϕ^0 are complex fields.

$$\begin{aligned} \phi^+ &= \frac{\phi_1 + i\phi_2}{\sqrt{2}} \\ \phi^0 &= \frac{\phi_3 + i\phi_4}{\sqrt{2}}. \end{aligned} \tag{54}$$

These two Higgs fields are related by a rotation in SU(2) space.

	I	I_3	Q	$Y = Q - I_3$
ϕ^+	$\frac{1}{2}$	$+\frac{1}{2}$	$+1$	$\frac{1}{2}$
ϕ^0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$

The most general renormalizable form of the scalar potential is

$$V = \mu^2 \phi\phi + \lambda(\phi\phi)^2. \tag{55}$$

Higher powers of ϕ lead to infinity in physical quantities. $V(\phi)$ is invariant under local gauge transformations. When $\mu^2 < 0$, we will have non-zero vacuum expectation value (v) of $|\phi|$:

$$\frac{v}{\sqrt{2}} = (\frac{-\mu^2}{2\lambda})^{1/2} \tag{56}$$

When a specific vacuum is chosen, that is, a preferred direction in weak isospin and hypercharge space is selected, symmetry breaks, *spontaneous symmetry breaking.*

Which one can we choose ? Since $\phi\phi = -\mu^2/(2\lambda) = v^2/2 = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)/2$, there are many ways to choose a direction. Let

$$\phi_1 = \phi_2 = \phi_4 = 0 \tag{57}$$

$$\phi_3 = v. \tag{58}$$

Thus Higgs doublet at the minimum potential is

$$\phi_0 = \frac{1}{\sqrt{2}}(0, v). \tag{59}$$

The neutral vacuum is required since the vacuum everywhere can emit or absorb a quantum of the Higgs field tht carries weak isospin and hypercharge. During this procedure, all quantum numbers of the Higgs field can vanish into the vacuum (absorbing the Higgs field) or create from the vacuum (emitting th Higgs field). If the vacuum is charged, the electric charge can not be conserved contrary to observation. Now, the normal precedure for handling perturbation is

1. find the minimum of the potential

2. expands the field around their value at the minimum and determines the exitations

3. in field theory, the ground state is vacuum, and excitations are particles.

$$\phi(x) = \frac{1}{\sqrt{2}}(0, v + h(x)) \tag{60}$$

We now need to look for equations satisfied by $h(x)$.

More general term for vecuum expectation value (v.e.v) will be

$$\phi(x) = exp[i\vec{\tau} \cdot \vec{\chi}(x)]\frac{1}{2}(0, v + h(x)) \tag{61}$$

where there are 4 fields, $\chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$, and $h(x)$. However, by a finite gauge transformation under SU(2), we can eliminate $\vec{\chi}(x)$ in the Lagrangian. It is called *"gauging away three fields"*.

They will reappear as the longitudinal components of W^+, W^- and Z when these bosons acquite mass : *"Three degrees of freedom have been eaten by the gauge fields."*

Let's examining the Higgs field term of the Lagrangian.

$$\begin{aligned}
D_\mu &= \delta_\mu - ieQA_\mu - \frac{ig}{\sqrt{2}}(\tau^+ Wk_\mu + \tau^- W^-_\mu) - ig_Z(\tau^3 - \sin^2\theta_W Q)Z_\mu \\
D\phi &= \frac{1}{\sqrt{2}}[\frac{ig}{\sqrt{2}}(v+h(x))W^+, \delta h(x) + \frac{ig_Z}{2}(v+h(x))Z] \\
L_\phi &= (D\phi)^2 - V(\phi^2)
\end{aligned}$$

$$\begin{aligned}
(D\phi)^2 &= \frac{1}{2}[(-\frac{ig}{\sqrt{2}}(v+h)W^-)(+\frac{ig}{\sqrt{2}}(v+h)W^+) + (\delta h - \frac{ig_Z}{2}(v+h)Z)(\delta h + \frac{ig_Z}{2}(v+h)Z)] \\
&= \frac{1}{2}[\frac{g^2}{2}(v+h)^2(W^+W^-) + (\delta h)^2 + \frac{g_Z^2}{8}(v+h)^2 ZZ] \\
&= \frac{1}{4}g^2v^2(W^+W^-) + \frac{g_Z^2}{8}v^2 ZZ \\
&+ \frac{1}{2}g^2 v(hW^+W^-) + \frac{1}{4}g_Z^2 v(hZZ) \\
&+ \frac{1}{4}g^2(hhW^+W^-) + \frac{1}{8}g_Z^2(hhZZ).
\end{aligned} \tag{62}$$

4.1 Mass : v^2 terms

The v^2 terms provide mass terms:

$$\frac{1}{4}g^2v^2(W^+W^-) + \frac{g_Z^2}{8}v^2 ZZ = M_W^2 W^+W^- + \frac{1}{2}M_Z^2 ZZ + \frac{1}{2}M_A^2 AA. \tag{63}$$

Thus

$$M_W = \frac{1}{2}gv \tag{64}$$

$$M_Z = \frac{1}{2}g_Z v = \frac{1}{2}\frac{g}{\cos\theta_W}v = \frac{M_W}{\cos\theta_W} \tag{65}$$

$$M_\gamma = 0 \tag{66}$$

In the electroweak theory, there are three free parameters and they are two gauge coupling constants g, g' and v. These parameters as inputs predict electric charge, M_W, M_Z, and G_F via the following relationships as we have shown in previous sections:

$$v = [\sqrt{2}G_F]^{-1/2} \tag{67}$$

$$\sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2} \tag{68}$$

$$= \frac{\pi\alpha_{em}}{\sqrt{2}G_F M_W^2} \tag{69}$$

$$= \frac{e^2}{g^2} \tag{70}$$

$$= 1 - \frac{e^2}{g'^2} \tag{71}$$

The theory is democratic so that any three variables in the above equations may be chose as inputs. It makes much more sense to choose the three parameters most accurately measured so that the predictions for any other measurements become as precise as possible, allowing sensitive tests of the theory. In the early 1980s, those were e (or α_{em}), G_F and the electroweak mixing parameter $\sin^2\theta_W$. G_F is extracted from the muon lifetime measurement:

$$G_F = 1.166 \times 10^{-5}. \tag{72}$$

Thus the vacuum expectation value is

$$v = 246 \text{ GeV}. \tag{73}$$

The neutrino scattering experiments measure $\sin\theta_W$. In 1978, $\sin\theta_W$ was measured to be around 0.23. From G_F, e, and $\sin\theta_W$, the W boson mass can be predicted:

$$M_W = \frac{1}{2}gv = \frac{1}{2}\frac{3}{\sin\theta_W}[\sqrt{2}G_F]^{-1/2} = \frac{37.42}{\sin\theta_W} \text{ GeV} = 78 \text{ GeV}. \tag{74}$$

$$M_Z = \frac{M_W}{\cos\theta_W} = 90 \text{ GeV} \tag{75}$$

This prediction lead to the discovery of the W and Z boson at $Sp\bar{p}S$ at CERN in 1982. By the end of 1989, the 4 LEP experiments measured M_Z to within 160 MeV. Since then α, G_F and M_Z were adopted as the basic input parameters.

4.2 Higgs - Gauge Boson Coupling terms

The hWW and hZZ terms represent the vertices with one Higgs field, W^+ and W^- and with one Higgs field, and two Z's:

$$L_{hWW,hZZ} = \frac{1}{2}g^2v(hW^+W^-) + \frac{1}{4}g_Z^2v(hZZ) = M_Wg(hW^+W^-) + \frac{1}{2}M_W\frac{g}{1-\sin^2\theta_W} \tag{76}$$

The $hhWW$ and $hhZZ$ terms represent the vertices with two Higgs field, W^+ and W^- and with two Higgs fields and two Z fields:

$$L_{hhWW,hhZZ} = \frac{1}{4}g^2(hhW^+W^-) + \frac{1}{8}g_Z^2(hhZZ) \tag{77}$$

As shown in the equations above, the couplings between the Higgs fields and Gauge boson fields are determined by purely SU(2) and U(1) couplings.

4.3 Higgs Mass

The Higgs mass terms comes from $V(\phi^2)$. However, there are no relations between the Higgs fields and other electroweak measurements: the Higgs mass is a free parameter to be measured.

4.4 Fermion Masses

Fermion-Higgs interaction terms which are invariant under SU(2) transpormations provide fermion mass terms. Here is an example for electrons :

$$\begin{aligned} L &= g_e(\bar{L}\phi e_R + \phi^+ \bar{e_R} L) & (78) \\ &= \frac{g_e v}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L) + \frac{g_e}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L)h & \\ &= \frac{g_e v}{\sqrt{2}}\bar{e}e + \frac{g_e}{\sqrt{2}}\bar{e}e & (79) \\ &= m_e \bar{e}e + \frac{m_e}{v}\bar{e}eh & (80) \end{aligned}$$