FINAL EXAM REVIEW 2

Problem : Non-degenerate Perturbation Theory

Consider a 1-D SHO. Calculate the shift in the ground state energy introduced if one uses the relativistic expression (correct to 1st order) for the kinetic energy.

Solution:

Recall from relativity that $T = E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

$$= mc^2\left[\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right] = mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \cdots - 1\right] = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

$$\therefore H' = -\frac{p^4}{8m^3c^2}$$

$|0\rangle$ is non-degenerate:

1st order correction is $\langle 0|H'|0\rangle = -\langle 0|\frac{p^4}{8m^3c^2}|0\rangle = -\frac{1}{8m^3c^2}\left(\langle 0|p^2\right)\left(p^2|0\rangle\right)$

recall that $p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^\dagger - a)$

$$p^2 = -\frac{\hbar m\omega}{2}\left(a^{\dagger 2} - aa^\dagger - a^\dagger a + a^2\right) = -\frac{\hbar m\omega}{2}\left(\sqrt{2}|2\rangle - |0\rangle\right)$$

$$\langle 0|H'|0\rangle = -\frac{1}{8m^3c^2}\left(\frac{\hbar m\omega}{4}\right)^2 \left(\sqrt{2}\langle 2| - \langle 0|\right)\left(\sqrt{2}|2\rangle - |0\rangle\right)$$

$$= -\frac{1}{8m^3c^2}\left(\frac{\hbar m\omega}{4}\right)^2\left(2+1\right) = -\frac{3(\hbar m\omega)^2}{32m^3c^2} = -\frac{3}{8}\frac{E_0^2}{mc^2}$$

Problem : Time Dependent Perturbation Theory:

1-D infinite well ($V=0$ $0<x<L$) contains a particle of mass m in the ground state. At time $t=0$, a perturbation $H' = A\delta(x-L/2)$ is turned on. Compute the initial ($\omega t \ll 1$) transition rate into 1st & 2nd excited states using time dependent perturbation theory.

Solution:

$$c_n(t) = \frac{1}{i\hbar}\int_{t_0}^{t} e^{i\omega_{nk}t'} H'_{nk}\, dt' \quad \text{where} \quad H'_{nk} = \langle n|H'|k\rangle \quad \text{initially in state } |k\rangle$$

$$\omega_{nk} = \frac{(E_n - E_k)}{\hbar}$$

$$c_2(t) = \frac{1}{i\hbar}\int_0^t e^{i\omega_{21}t'} \underbrace{\int_0^L \frac{2}{L}\sin\left(\frac{2\pi x}{L}\right) A\delta(x-L/2)\sin\left(\frac{\pi x}{L}\right) dx}_{\frac{2A}{L}\sin\left(\frac{2\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 0} dt'$$

$$= 0. \Rightarrow R_{1\to 2} = 0.$$

$$c_3(t) = \frac{1}{i\hbar}\int_0^t e^{i\omega_{31}t'} \underbrace{\int_0^L \frac{2}{L}\sin\left(\frac{3\pi x}{L}\right) A\delta(x-L/2)\sin\left(\frac{\pi x}{L}\right) dx}_{\frac{2A}{L}\sin\left(\frac{3\pi}{2}\right)\sin(\pi/2) = -\frac{2A}{L}} dt'$$

$$= -\frac{2A}{iL\hbar}\int_0^t e^{i\omega_{31}t'} dt' \qquad \omega_{31} = \frac{\hbar^2\pi^2(9-1)}{2mL^2} = \frac{4\pi^2\hbar^2}{mL^2} = \omega$$

$$\int_0^t e^{i\omega t'} dt' = \frac{1}{i\omega} e^{i\omega t'}\Big|_0^t = \frac{1}{i\omega}\left(e^{i\omega t} - 1\right) = \frac{e^{i\omega t/2}}{i\omega}\left(e^{i\omega t/2} - e^{-i\omega t/2}\right) = \frac{2e^{i\omega t/2}}{\omega}\sin\left(\frac{\omega t}{2}\right)$$

$$\therefore P_{1\to 3}(t) = |c_3(t)|^2 = \frac{4A^2}{L^2\hbar^2}\frac{4}{\omega^2}\sin^2(\omega t/2) = \frac{16A^2}{L^2\hbar^2\omega^2}\sin^2(\omega t/2)$$

$$R_{1\to 3}(t) = \frac{dP}{dt} = \frac{16A^2}{L^2\hbar^2\omega^2}\, 2\cdot \underbrace{\sin\left(\frac{\omega t}{2}\right)}_{\approx \frac{\omega t}{2}} \underbrace{\cos\left(\frac{\omega t}{2}\right)}_{\approx 1} \frac{\omega}{2} = \frac{8A^2}{\hbar^2 L^2} t$$

Problem : Sudden Approximation

A particle is in the ground state of a one dimensional infinite square well with walls at $x=0$ & $x=L$. At time $t=0$ the walls are suddenly removed so that the particle becomes free.

(a) Find the probability $\pi(p_x)dp_x = |\phi(p_x)|^2 dp_x$ that a measurement of the momentum of the particle will produce a result between p_x & $p_x + dp_x$

(b) Explain why a measurement of the energy of the particle after the walls are removed need not give the ~~same~~ same result as the initial energy.

Remember that for a free particle $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{ikx} dx$

continuum analog of

$$\phi(x) = \sum_k d_k \phi_k(x)$$

Prob of being in an eigenstate is $|d_k|^2 \to |\phi(k)|^2$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x,0)\, e^{-ikx} dx$$

same thing as $d_k = \sum_n c_n \langle \phi_k | \psi_n \rangle$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{\pi x}{L}\right) e^{-ikx} dx = \sqrt{\frac{1}{\pi L}} \frac{1}{2i} \int_0^L e^{i(\frac{\pi x}{L} - kx)} - e^{-i(\frac{\pi x}{L} + kx)} dx$$

$u = i(\frac{\pi}{L} - k)x$

$du = i(\frac{\pi}{L} - k)dx$

$$= \frac{1}{2i}\sqrt{\frac{1}{\pi L}} \left[\frac{1}{i(\pi/L - k)} \left(e^{i(\pi/L - k)L} - 1\right) + \frac{1}{i(\pi/L + k)} \left(e^{-i(\pi/L + k)L} - 1\right) \right]$$

$$= -\frac{1}{2}\sqrt{\frac{1}{\pi L}} \left[\frac{L}{\pi - kL} \underbrace{e^{i(\pi - kL)}}_{-e^{ikL}} + \frac{L}{\pi + kL} \underbrace{e^{-i(\pi + kL)}}_{-e^{-ikL}} - \left(\frac{L}{\pi - kL} + \frac{L}{\pi + kL}\right) \right]$$

Recall that $e^{\pm i\pi} = -1$

$$= -\frac{1}{2}\sqrt{\frac{\hbar}{m}}\left[\frac{-2\pi L e^{-ikL} - 2\pi L}{\pi^2 - (kL)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{m}}\,\frac{2\pi L}{\pi^2 - (kL)^2}\left[1 + e^{-ikL}\right] = \frac{2\sqrt{\pi L}}{\pi^2 - (kL)^2}\, e^{ikL/2}\cos(kL/2)$$

Square to get desired result.

(b) Although $\sin\left(\frac{n\pi x}{L}\right)$ is an energy eigenstate of $H = \frac{p^2}{2m}$. It is only so w/ the specific boundary conditions i.e $H=0$ outside of $0<x<L$.

True eigenstates of $\frac{p^2}{2m}$ are $\frac{1}{\sqrt{2\pi}} e^{ikx}$ ~~which can be w..~~

$\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ can be written in terms of these eigenstates $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(k)e^{ikx}$

& can give a ~~[illegible]~~ different energies w/ prob $|\phi(k)|^2$.